Extended Essay- Mathematics

Distribution of Identical Objects

"How can the ways of distributing identical objects be used in solving various algebra problems?"

Word Count: 2901

Contents:

1. Introduction:

Mathematics is a science that includes logic of shape, quantity and arrangement.[1] It is related with almost everything in life. From engineering and economics to music and art mathematics is all around us. Mathematics is divided into seven different branches: algebra, calculus and analysis, geometry and topology, combinatorics, logic and number theory.[2] Areas of mathematics are not independent from each other. A problem might not be solved using only one way; it can be solved by using a various ways and various areas of mathematics.

There are lots of problem solving techniques in mathematics and each technique does not belong to one specific area. In this extended essay, I will investigate one of the famous problems and its solution known as "Distribution of identical balls to distinct boxes" related to combinatorics. Combinatorics is related with algebra, graph theory and many other areas. However, it is primarily concerned with counting the number of ways in which a finite sequence of operations can be performed and how discrete objects can be arranged or selected. [3] In this essay, after explaining the solution method, I will use this method to solve some algebraic problems that are frequently asked in university entry examination in my country and Mathematics Olympiads. The method will applied to problems related to equations, inequalities multinomial expansion, positive divisors of a natural number and real life application. Therefore my research question for this thesis is;

Research Question: "How can the ways of distributing identical objects be used in solving various algebra problems?"

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[1] "What is Mathematics?".Live Science.web. https://www.livescience.com/38936-mathematics.html.17.02.2019
[2]"Life After Calculus".Cornell University.web.https://math.cornell.edu/life-after-calculus.17.02.2019
[3] Öztürk, Fikri. *Combinatoric*. Ankara. 1995

To answer this question the formulas and explanations required for permutation, combination and repeated permutation are given firstly. All of these formulas are essential for generalizing the formulas that are required for distributing identical objects. By solving these questions, I realized that the logic behind each question is to think of what is given in questions as objects and distributing them to distinct boxes. This allowed me to construct my hypothesis:

Some algebra problems can be solved by adapting the concept to identical objects and distinct boxes and using suitable "Distribution of Identical Objects" formulas.

My primary reason for investigating distribution of identical objects as my extended essay topic is that combinatorics has high range of applicability to other areas of mathematics as well as real life problems. Therefore, there are a lot of things to investigate about combinatorics. In distribution of identical objects, there are some formulas about distributing identical objects to distinct boxes which can be an interesting topic if it is applied to other areas of mathematics.

To begin with, this extended essay's topic lies within three main areas in combinatorics; combination, permutation and repeated permutation.

Permutation is an arrangement of a number of objects in a definite order.[4] Arrangement of "n" objects taking "r" at a time can be shown as $P(n,r)$. The formula for permutation comes from the following arrangement. Consider "n" objects to choose from and we choose "r" of them where $n \geq r$;

$$\frac{n}{1^{st}\ place} \cdot \frac{(n-1)}{2^{nd}\ place} \cdot \frac{(n-2)}{3^{rd}\ place} \cdot \frac{(n-3)}{4^{th}\ place} \cdot \quad \dots \quad \cdot \frac{(n-r+1)}{r^{th} place}$$

[4] Öztürk, Fikri. *Combinatoric*. Ankara. 1995

The lines below the number of objects represent the places where the objects are put. The mathematical expression for the explanation above is showed below:

$$n.(n-1).(n-2).(n-3).\ldots.(n-r+1) = \frac{n!}{(n-r)!}$$

Hence, the formula for permutation is

$$\frac{(Total\ number\ of\ objects)!}{(Total\ number\ of\ objects - the\ number\ of\ objects\ to\ be\ arranged)!}$$

$$P(n,r) = \frac{n!}{(n-r)!}$$

Combination is selecting a number of objects in any order.[5] Selecting "r" objects at a time from "n" objects can be shown as $C(n,r)$. Apart from Permutation, order does not matter in combination. The difference between permutation and combination formula is "$r!$". Since "$r!$" is the number of ways arranging "r" objects, the proof for combination is:

$$C(n,r).r! = P(n,r)$$

$$C(n,r) = \frac{P(n,r)}{r!}$$

When we write the formula for permutation to its place:

$$C(n,r) = \frac{n!}{(n-r)!}.\frac{1}{r!}$$

[5] Öztürk, Fikri. *Combinatoric*. Ankara. 1995

Hence the combination formula occurs.

$$C(n,r) = \frac{n!}{(n-r)!.\,r!}$$

Repeated permutation is used when finding the number of different permutations of “n” objects where the first objects appears“n_1” times, the second object appears “n_2” times and “k^{th}” object appears “n_k” times. Since changing places of same objects does not affect the order, we can calculate total number of permutations by;

$$\frac{n!}{n_1!.\,n_2!.\,\dots.\,n_k!}$$

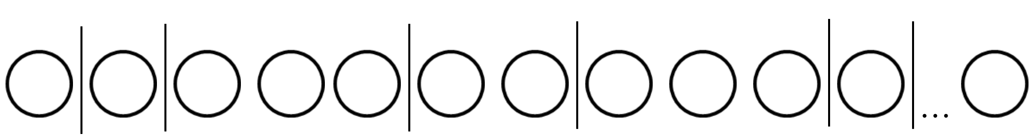
Where “$n = n_1 + n_2 + \cdots + n_k$”

2. Principles of Distribution of Identical Objects:

Combination, permutation and repetitive combination explanations mentioned before are for distinct objects. When repetitive combination formula is applied to identical objects, this situation can be expressed in three sections.

2.1. Distribution of “n” identical objects to “r” boxes without a special case:

The figure above shows “n” objects. To distribute this “n” objects, “$r - 1$” separators must be put between the objects in order to divide them into “r” groups.



This can be coded by using “0” to represent the objects and “1” to represent the separators between them. As an example, the figure above is coded.

$$0\ 1\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 1\ \ldots\ 0$$

Since the objects and the separators are identical, they are repeated. The expression previously stated for repeated permutation can be applied to this situation. Therefore,

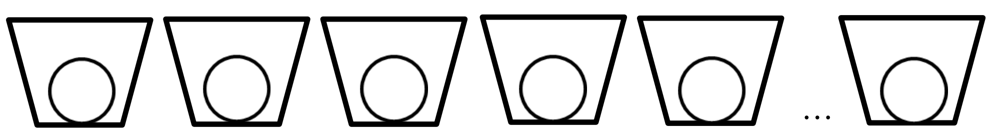
$$\frac{(n+r-1)!}{n!.\,(r-1)!}$$

is obtained. By definition, it is equal to;

$$C(n+r-1, r-1)$$

2.2. Distribution of identical objects putting at least one object in each box:

In this case, the number of identical objects must be greater than the number of boxes $(n \geq r)$. The first step is to put one object in each box which makes “r” objects in total.



Then replace the remaining objects which is in equipoise with the distribution of"$n-r$” objects into “r” boxes, them similar to the first method in distribution of identical objects. This actively illustrates that, “$r-1$” separators should be put between “$n-r$” objects. This method can be used in similar cases such as putting at least two objects in each box

Hence when combination formula and the first principle of distribution of identical objects applied to "$n-r$" objects and "r"boxes the number of distribution formula is;

$$C(n-r+r-1, r-1) = C(n-1, r-1)$$

$$= \frac{(n-1)!}{(r-1)!\,.\,(n-r)!}$$

This is also equal to$C(n-1, n-r)$.

2.3 Formulizing the formula required for any special cases:

If we formulize this method for putting at least "z_1" objects in the first box, "z_2" objects in the second box to "z_r" objects in the "r"th box from "n" objects (where $n \geq \sum_{i=1}^{r} z_i$);

$1st\ Box \geq z_1$ $\quad$ $2nd\ Box \geq z_2$ $\quad$ $3rd\ Box \geq z_3$ $\quad$ $rth\ Box \geq z_r$

First, $\sum_{i=1}^{r} z_i$ objects are distributed to "r" boxes similar to the method stated in 2.2. Then, $n - \sum_{i=1}^{r} z_i$ objects which are remained are distributed into "r"boxes. The expression for this is:

$$C\left(n - \sum_{i=1}^{r} z_i + r - 1, r - 1\right) = \frac{n - \sum_{i=1}^{r} z_i + r - 1}{(r-1)!}$$

3. Applications

1. This question is about equations. The equation contains three unknowns which will be thought as distinct boxes and the value will be the objects. The formula in 2.1 is used to solve the section "i" and the formula in 2.2 is used to answer "ii".

   How many different (x, y, z) values which satisfies the equation $x + y + z = 10$ for;

   i. x, y, z are natural numbers

   ii. x, y, z are positive natural numbers.

   Solution:

i. To solve this question, x, y and z should be imagined as different boxes and the equation should be thought as 10 balls distributing in three different boxes. Since there are three boxes, two separators are required.

When the question is applied to the formula for distribution of identical objects given in section 2.1, the result is obtained.

$$C(n + r - 1, r - 1) = \frac{(n + r - 1)!}{n!\,.\,(r - 1)!}$$

$$C(12\,,2) = \frac{12!}{10!\,.\,2!}$$

The answer is:

$$\frac{12!}{10!\,.\,2!} = 66$$

ii. In positive natural numbers, each box in section i, cannot have zero balls. There has to be at least one ball in each box. Therefore, I used the formula that is explained in 2.2 to find the result.

$$C(n-1, r-1) = \frac{(n-1)!}{(r-1)!\,.\,(n-r)!}$$

$$C(9,2) = \frac{9!}{7!\,.\,2!} = 36$$

2. This question is about inequalities. There are four unknowns in the inequality below which will be thought as boxes and the value in the equality will represent balls. A new box is added to solve this question and the method for solving it is stated in 2.1.

For $x, y, z, t \in N$ how many (x, y, z, t) values satisfying the inequality

$$x + y + z + t \leq 20$$

Solution:

x, y, z, t should be thought as distinct boxes similar to the previous question. I will add another distinct box to it represented with the letter "w". This way, when 20 objects are distributed to five boxes, the excess box I created will contain the balls which are unused. Since there are five boxes, four separators are needed to use the formula stated in 2.1.

$$C(n+r-1, r-1) = \frac{(n+r-1)!}{n!.\,(r-1)!}$$

$$C(24,20) = \frac{(20+4)!}{20!\,.\,4!}$$

The result is:

$$\binom{24}{20} = 10\ 626$$

3. This question is about binomial and multinomial expansions. In binomial expansions there are terms formed with sum of two terms with an exponent on the sum as shown; $(x + y)^n$. The total number of terms in binomial expansion is found by "$n + 1$". However, there are four unknowns in the multinomial expansion below. Like binomial expansions, the sum of the exponents of the unknowns will give the exponent of the multinomial expansion. However, the total number of terms can be found by using distribution of identical object formulas. This question is solved by using the formula in 2.1

How many different terms in the multinomial expansion of $(x + y + z + t)^{20}$

Solution:

In multinomial expansions, there is a coefficient in each term and the sum of exponents of the terms is equal to the given equation's exponent. For example, in this question let the expression below be a term of the multinomial expansion given in the question.

$$k(x^a.\,y^b.\,z^c.\,t^d)$$

Where "k" is the coefficient and $a + b + c + d = 20$

Using the formula for distribution of identical objects in 2.1 where a, b, c, d are distinct boxes and 20 is the number of identical objects.

$$C(n+r-1, r-1) = \frac{(n+r-1)!}{n!.(r-1)!}$$

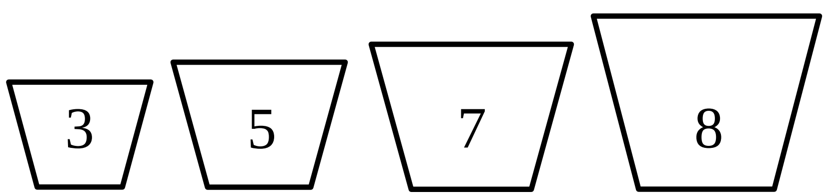
$$C(23,3) = \frac{23!}{20!\ .(3)!}$$

$$= 1{,}771$$

4. This question is about an application of distribution of identical objects. There is a condition stated in the question below. To provide that condition, the method which is stated in 2.3 is used.

How many different ways can 19 identical balls are distributed into four boxes which can contain at most 3, 5, 7 and 8 balls respectively?[6]

Solution:



These four boxes can contain at most twenty three balls in total. This means

"$23 - 19 = 4$" there will be four empty spaces in these four boxes. To solve this question, the empty spaces should be distributed to four boxes. Therefore, we imagine the spaces as four

[6] National Math Olympiads, 1999

identical balls and distribute them to four boxes. To divide four balls into four groups we need to put three separators. Using the combination formula stated for without any special cases we obtain,

$$C(n+r-1, r-1) = \frac{(n+r-1)!}{n!\,.\,(r-1)!}$$

For $n = 4$ and $r = 4$ respectively. Hence;

$$C(7,3) = \frac{7!}{4!\,.\,3!} = 35$$

However, the first box can contain at most three balls so it cannot have 4 empty spaces inside. Therefore the situation which is $(4,0,0,0)$ should be subtracted from the result of the combination. Thus the answer is;

$$35 - 1 = 34$$

5. This question is about inequality with a condition. This question is answered considering the condition stated below and using the formula stated in 2.1.

   How many (A, B, C, D) values that satisfy the equation $A + B + C + D = 30$ such that $A + B > C + D$?[7]

   Solution:

   A and B can take values from sixteen to thirty while C and D can take values from fourteen to zero.

[7] National Math Olympiad, 1999

The table shows the probability of sum of balls according to the case given in the question.

A and B	C and D
16	14
17	13
18	12
19	11
20	10
21	9
22	8
23	7
24	6
25	5
26	4
27	3
28	2
29	1
30	0

These objects can be located in different ways inside A and B or C and D. To find the different ways of these selections in each pair of boxes, the formula for distribution of identical objects without a special case is applied to these probabilities. For example;

A and B are two boxes. 16 balls can be located in these two boxes. Thus "$n = 16$" and "$r = 2$" in $C(n + r - 1, r - 1)$ formula which equals to; $\frac{17!}{16!.1!}$. For this probability, the sum of balls in C and D equals to 14. Thus "$n = 14$" and "$r = 2$" is applied to the previous formula and $\frac{15!}{14!.1!}$ is obtained. The product of these two expressions gives the different distribution for this probability.

$$\frac{17!}{16!.1!} . \frac{15!}{14!.1!} = 17 . 15$$

When all of these steps are repeated for different probabilities the expression below is obtained:

$$\binom{17}{16}.\binom{15}{14}+\binom{18}{17}.\binom{14}{13}+\binom{19}{18}.\binom{13}{12}+\cdots+\binom{31}{30}.\binom{1}{0}$$

Since it is combination, it equals to the expression below:

$$17.15+18.14+19.13+20.12+\cdots+31.1$$

This expression can be rewritten as;

$$\sum_{k=1}^{15}(k+16).(16-k)=\sum_{k=1}^{15}(256-k^2)$$

Which equals to;

$$256.15-\frac{15.16.31}{6}=2600$$

When I thought of this problem again, I found a shorter way to solve it:

$$A+B+C+D=30$$

Therefore there are;

$$\binom{33}{30}=5\,456$$

Different (A, B, C, D) values without any condition. If $A+B=C+D=15$, there are;

$$\binom{16}{1} = 16$$

Different values for both (A, B) and (C, D). When they are multiplied 256 different (A, B, C, D) values that satisfy $A + B = C + D = 15$ is obtained. When 256 is subtracted from the total different (A, B, C, D) values without any condition and the number obtained with this subtraction is divided by 2, the question is solved. The reason for dividing the result by "2" is there are two possible conditions for (A, B, C, D) values: either$A + B > C + D$ or $A + B < C + D$. Since both inequalities have same number of different values for (A, B, C, D) and both of them are included in the subtraction, to obtain the right condition for the question, the subtraction is divided by "2".

$$\frac{5\,456 - 256}{2} = 2600$$

6. This question is about an equation which contains three multiplied unknowns. To solve it, the given value is divided into its prime factors and the method stated in 2.1 is applied.

   How many (x, y, z) values possible which obtains the equation $x\ .\ y\ .\ z = 200000$ where $x, y, z \in N$? [8]

   Solution:

   $x\ .\ y\ .\ z = 200000$ can be rewritten as $x\ .\ y\ .\ z = 2\ .\ 10^5$

[8] National Math Olympiads,2005

Since x, y and z are natural numbers, they are equal to;

$$x = 2^{a_1}.5^{b_1}$$

$$y = 2^{a_2}.5^{b_2}$$

$$z = 2^{a_3}.5^{b_3}$$

Where a_1, a_2, a_3, b_1, b_2 and b_3 are not negative integers. Using the equation given in the question, we obtain $a_1 + a_2 + a_3 = 6$ and $b_1 + b_2 + b_3 = 5$. The first equation will have;

$$C(6+3-1,3-1) = \frac{(6+3-1)!}{\big(6+3-1-(3-1)\big)!\,.\,(3-1)!}$$

$$= \frac{8!}{6!\,.\,2!}$$

$= 28$ Solutions.

The second equation will have;

$$C(5+3-1,3-1) = \frac{(5+3-1)!}{\big(5+3-1-(3-1)\big)!\,.\,(3-1)!}$$

$$= \frac{7!}{5!\,.\,2!}$$

$= 21$ Solutions.

Since two equations are obtained, the product of the results gives the final answer. Therefore, there are $28\,.\,21 = 588$ different (x, y, z)values possible.

7. I created this question when I was thinking about what to eat before my course starts. I go to a course for university exam after school. Therefore, I feel hungry before the lesson starts in the course. I bought 10 cookies and I have to eat at least one cookie per day to suppress my hunger. How many different ways I can eat all of the cookies.

Solution:

This question is related with the formula for distribution of "n" identical objects putting at least a certain value in "1" boxes which is stated in 2.2.

If I finish my cookies in one day, there is only one solution which is: $C(9,0)$

If I decide to finish my cookies in two days, the solution is: $C(9,1)$

If I decide to finish my cookies in three days, the solution is: $C(9,2)$

These calculations will go on until I have the statement for the longest period of time which I can finish all of my cookies eating at least one cookie every day.

If I finish 10 cookies in 10 days, $C(1,6)$ is obtained- one possible situation. The sum of all these probabilities above gives the total different ways of eating my cookies.

$$\binom{9}{0}+\binom{9}{1}+\binom{9}{2}+\binom{9}{3}+\binom{9}{4}+\binom{9}{5}+\cdots+\binom{9}{9}$$

Which is also equal to $2^9 = 512$ ways to eat my cookies

8. This question is related with the positive divisors. Our method gives creative solution to find some positive divisors with given condition.

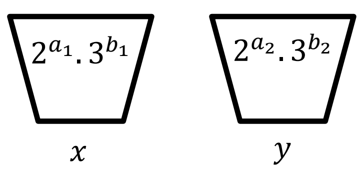
Let S be the set of all positive divisors of $A = 864$. How many ordered pairs (x, y) exists such that $x, y \in S$, but $x\,.\,y \notin S$

Solution:

Since $864 = 2^5\,.\,3^3$ then the number of positive divisors are $4\,.\,6 = 24$. Since two unknowns, x and y exists, there are 24 possibilities for each one so $24\,.\,24 = 576$ pairs of (x, y) exists.

Let's find the number of pairs which satisfies $x\,.\,y \in S$;

Imagine x and y as two distinct boxes. x contains $2^{a_1}.\,3^{b_1}$ objects and y contains $2^{a_2}.\,3^{b_2}$ objects simulated in the figure above;



$$a_1 + a_2 \leq 5$$

Which means distributing 5 identical balls to 3 boxes

$$\binom{7}{2} = 21$$

$$b_1 + b_2 \leq 3$$

Which means distributing 3 idenctical balls to 3 boxes

$$\binom{5}{2} = 10$$

Therefore $21.\,10 = 210$ (x, y) exist such that $x\,.\,y \in S$

For the next condition: $x, y \in Z^+, x$ and y divides 864 but $x \,.\, y$ does not, the values of $x \,.\, y$ that divides 864 are subtracted from the number of positive division pairs.

$$576 - 210 = 366 \text{ pairs}$$

4. Conclusion

In the algebra questions that I have solved, the formulas required for distribution of identical objects has applied to solve them more practically. Counting the possibilities for a certain question is also a possible way to solve it. However, the formulas that I have been used decreases the risk of mistakes as well as decreasing the time required for solving a certain question. Indeed, these formulas offer a simpler perspective to solve even complex algebra questions with combinatorics. When I solve the questions, I thought it is important to realize that the technique for solving these questions was to liken the concept to distribute a certain number of balls to different boxes. For example, in the fourth question I imagined empty spaces as objects.

Mastering the distribution of identical objects can be very useful from this aspect. Apart from that, it can also be used in solving real-life mathematical problems such as calculating how many different ways to replace an object or group of objects. From this reason, it can be an interesting topic for developing new ideas inspired by these problems.

When I was writing my thesis, I learned to look at the questions that I solve daily in question banks or in Olympiads with a different perspective and to have distinct comments. This also improved my way of thinking in solving questions. I started to question more and search for some practical ways of solving a certain question which increased my speed in solving questions as well as encouraged me to find extraordinary ways to solve a problem. At the beginning, I found it hard to adapt the questions to a different method. I extended the solutions of some problems like the fifth question. However, when I rethought of it, I found an easier way. Also it was extreme for me to explain each question thoroughly. In some questions, I was confused to define which one of terms should be liken to the box or the ball. Despite each difficulty, I believe that this extended essay was one of the biggest experiences of mine and it definitely improved me on mathematics.

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