INTERNATIONAL BACCALAUREATE PHYSICS EXTENDED ESSAY

WHAT IS THE RELATION BETWEEN SPECIFIC IMPULSE OF ROCKET ENGINES AND CHANGE IN MECHANICAL ENERGY OF THE ROCKET INDUCED BY ENGINE THRUST?

Word Count: 4000

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Introduction

Specific impulse describes the efficiency of a rocket or a jet engine, by itself. It is a required component of Tsiolkovsky rocket equation [(1)], also known as the classical rocket equation, which is used to calculate the delta-v (Δv) of a rocket. Delta-v is the denotation of change in velocity induced by a rocket engine and is vital for especially outer space missions since a space vessel (a probe, a satellite, etc.) escapes the gravitational field of a celestial body under the condition of its delta-v reaching or exceeding the escape velocity of the said celestial body, such as Earth. Other than displaying the efficiency of a rocket engine, specific impulse of the rocket is necessary for the calculations for deep space missions, i.e. lunar (Apollo missions) and interplanetary missions (Voyager).

I chose to explore the specific relation between rocket engine efficiency and the change in the total energy of the rocket caused by the engine because I believe having access to ready estimates of the specific impulse required to complete a certain task would greatly aid the designing process of rockets which are to perform orbital maneuvers.

Background Information

Specific impulse (I_{sp}) of a rocket is, by definition, the total impulse delivered per unit mass of propellant ejected. Specific impulse is used to describe the fuel consumption efficiency of chemical, ion, nuclear rocket and jet engines. Mathematically, I_{sp} is expressed with two different equations, each giving out values with different units. One of these expressions is the force of thrust exerted by the engine (F) divided by the mass flow rate of the propellant ($\dot{m}$), which gives out the unit of velocity ($m\,s^{-1}$). The more used expression divides the previous by the standard gravity (g_0), to transform mass flow rate into weight flow rate.

The I_{sp} equation is denoted as follows;

$$I_{sp} = \frac{F}{\dot{m} \cdot g_0} \left(\frac{kg \cdot ms^{-2}}{kgs^{-1} \cdot ms^{-2}} = s \right) \quad (2)$$

where specific impulse equals the force of thrust divided by the weight flow ratio of the engine, giving out the value with the unit of time (s).

The value of specific impulse of rocket engines is dependent on the propellant flow outwards through the engine nozzle, provided that the force of thrust is constant. Mass flow rate, the rate at which propellant is ejected from the engine nozzle, is affected by a multitude of factors; the cross-sectional area of the propellant outlet, pressure, temperature and velocity. During launch, all factors other than the cross-sectional area of the outlet are subject to change. Pressure decreases as the atmosphere is left, temperature decreases overall as outer space is reached and velocity increases continuously due to the constant exertion of thrust. Due to the combined effects of changes in these factors, especially rockets designed for space flight, see an increase in their I_{sp} values as they gain altitude. Once outer space is reached, the systems of the engine are in an internal equilibrium with no exterior factors affecting the mass flow rate. Only the change in velocity is prevalent in outer space travel, which has negligible effects on the mass flow rate; hence the specific impulse values are stabilized. Therefore, in order to prevent complications caused by the recursive change in I_{sp} during vertical atmospheric flight, where increase in altitude and velocity increases I_{sp} which causes a higher increase in altitude and velocity in return, the experiment is held in Low Earth Orbit.

Engine thrust delivers impulse which changes the total energy of the spacecraft. In orbital mechanics, an object in orbit of a body has a total amount of energy specific to its orbital radius and mass. If the craft is pointed towards the prograde-direction during its burn, the direction at which the orbiter moves in orbit, energy is added to the total energy of the craft. A burn in the opposite direction (retrograde) kills orbital velocity, decreasing its total energy.

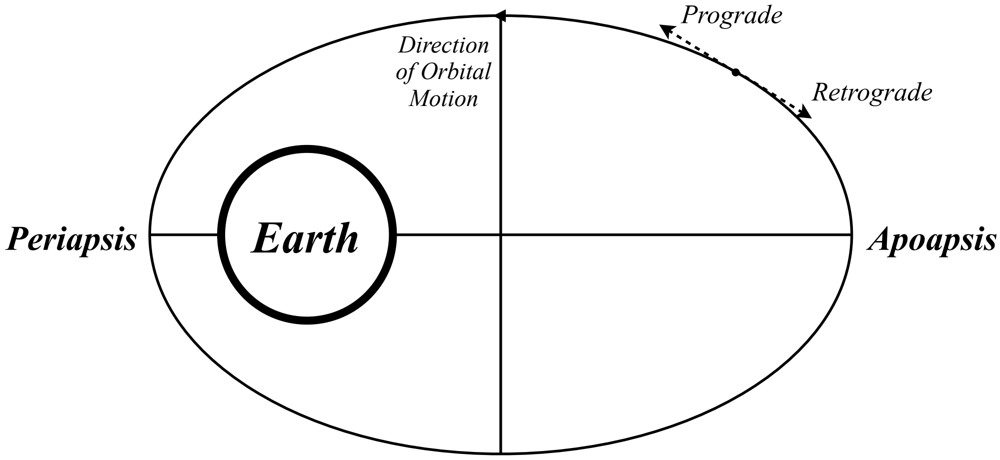


Figure 1 *Elliptical Orbit*

Increasing the total energy of an orbiter through thrust increases the average radius of its orbital path, which is done so by pulling the periapsis, the lowest point in orbit, closer to the location of the orbiter at the time of the burn and extending the apoapsis, the highest point in orbit, farther away. As both of these points increase in altitude, the average orbital radius increases. This method of modifying orbits and mechanical energy calculations are used primarily in Hohmann transfers, an orbital maneuver to transfer the orbiter from one circular orbit to another by burning at least two times during its procedure.

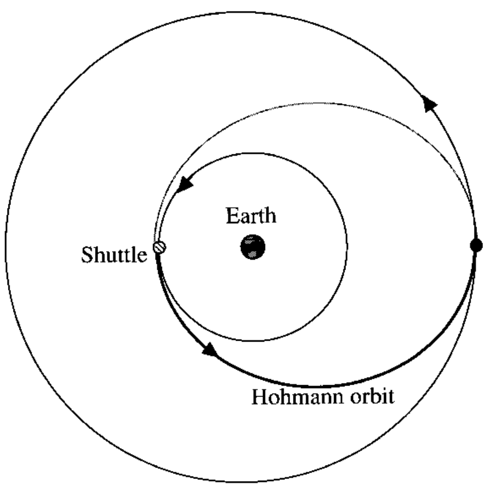


Figure 2 *Hohmann Transfer* (3)

The intermediate orbit between two circular orbits, the Hohmann orbit, (as seen in **Figure 2**) is the result of a singular burn in the prograde-direction, at the apoapsis of which the second circularizing burn takes place. The first burn increases the total energy transforming the initial orbit into a Hohmann orbit. The equation of total mechanical energy in circular orbit, which is used to calculate the initial energy of the orbiter in its initial orbit, is as follows;

$$E_m = \frac{-GMm}{2R} \ (J) \ ^{(4)}$$

where the mechanical energy of the orbiter (E_m) equals the negative of the product of universal gravitational constant ($G/m^3kg^{-1}s^{-2}$), mass of the body being orbited (M/kg) and mass of the orbiter (GM is also denoted as μ/m^3s^{-2}), divided by the orbital radius (R/m) multiplied by two. The reason for this equation being negative is that work done to bring an object of mass *m* to a circular orbit with the radius of *R* around a mass of *M*, no work needs to be done. In fact the system does work by itself, hence the negative sign in the equation.

Change in mechanical energy induced by rocket engine during its burn, dependent variable of this experiment, equals the difference between the mechanical energies of the initial and modified (Hohmann) orbits, as this change in energy is due to thrust.

While initial mechanical energy prior to the burn can be calculated by the previous equation, since initial orbit is circular, mechanical energy of the modified Hohmann orbit cannot be determined with the same equation. The reason why is because as fuel is burned, periapsis stays approximately at the initial distance from the center, as the location of the orbiter during the burn is irrelevant due to the stable radii of circular orbits, apoapsis gains altitude, resulting in an elliptical orbit as seen in **Figure 2**.

Mechanical energy, by definition, denotes the total energy of an object which consists of two different energy types: kinetic and gravitational potential energies. In elliptical orbital motion, mechanical energy of a satellite is conserved as long as there is no exterior disturbance or orbital decay, provided that mass stays constant after burn. This conservation is satisfied by the state of balance between gravitational potential energy (E_g) and kinetic energy (E_k). Velocity, therefore E_k, of orbiter increases as its altitude, and consequently its E_g decreases, whilst the total sum of its energy stays stable.

In the case of an elliptical orbit, such as a Hohmann orbit, because of the aforementioned energy balance, where the gravitational potential energy of a satellite orbiting on such a path is the least, the kinetic energy is the greatest. Therefore, at the periapsis, the point at which the satellite is closest to the body it orbits (Earth), kinetic energy is the greatest while gravitational potential energy is the smallest. In order to be able to calculate the mechanical energy of the orbiter in a Hohmann orbit, both the gravitational potential and kinetic energies are required. The gravitational potential energy equation of a satellite with known orbit radius is as follows;

$$E_g = \frac{-GMm}{R}\ (J)$$ [4]

where R denotes the distance between the center of mass of the body responsible of gravitational pull, and the satellite. This formula has a minus sign alike the previous mechanical energy equation for the same reason. Mechanical energy of a satellite in an elliptical orbit is calculable when gravitational potential energy calculated is added to kinetic energy of the orbiter, the equation of which is as follows;

$$E_k = \frac{mv^2}{2}\ (J)$$ [4]

****** The data for the calculation of the energies, namely the orbital radius, velocity and mass, are measured at the apoapsis.***

Despite the purpose of real engines, a zero-I_{sp} engine, where no force is exerted, should also satisfy the conclusion. A zero-I_{sp} engine only reduces the mass of orbiter, by ejecting propellant with no force output. Although mass reduction without thrust would not affect the orbit, E_m of orbiter would decrease with mass. The calculation of change in mechanical energy due to sole mass reduction would give out the smallest practical value of ΔE_m under given circumstances —orbital radius and propellant mass.

On the other hand, where rocket engine is more efficient, rocket engine may deliver enough impulse to reach escape velocity even with low amounts of fuel. Reaching escape velocity means that the rocket will be able to just escape the gravity well of the body. In other words, the rocket will have zero velocity at infinity once it leaves the gravitational field of influence of the planet. The total energy of the rocket will equal to zero at infinity, meaning that the change in mechanical energy equals the initial total energy of the rocket as it is nullified by the engine thrust. The velocity required to escape a body can be calculated as follows;

$$v_e = \sqrt{\frac{2GM}{R}} \; (m s^{-1}) \; ^{(5)}$$

However, the initial orbital velocity should also be considered as the velocity due to engine thrust will add up to it. I_{sp} value required for any given rocket with a specific orbital velocity and propellant mass to reach a certain velocity can be calculated via the following Delta-v equation;

$$\Delta v = v_e - v_i = ln\left(\frac{m_i}{m_f}\right) \cdot I_{sp} \cdot g_0 \ ^{(1)}$$

where the change in velocity (Δv) required to escape equals the natural logarithm of the ratio of initial mass (m_i) of the rocket to the final mass (m_f), multiplied by the specific impulse value and standard gravity.

Any engine with a higher I_{sp} value than the value required for escape velocity will result in velocity at infinity, thus kinetic energy, implying that the engine induces a larger change in energy. This remnant velocity is called hyperbolic excess velocity, the equation of which is as follows;

$$v_\infty = \sqrt{\frac{-GM}{a}}\ (ms^{-1})\ ^{(5)}$$

where a denotes the semi-major axis of the orbital path which, in this case, is hyperbolic. The equation for semi-major axis is as follows;

$$a = \frac{1}{(\frac{2}{R}) - (\frac{v^2}{GM})}\ (m)\ ^{(5)}$$

At any specific impulse value greater than the required for escape velocity, the final mechanical energy of the rocket at infinity consists solely of kinetic energy due to hyperbolic excess velocity.

The difference between the initial mechanical energy and the sum of the final kinetic and gravitational potential energies (final mechanical energy) induced by the rocket engine during its orbital maneuver burn, can be related to engine efficiency, the specific impulse. At the same rate of fuel consumption with the same amount of fuel, different engines with different I_{sp} values are to perform differently, delivering disparate levels of impulse and therefore having distinct orbits they can reach. If clarified, such a relation would be beneficial for the selection of engines used in rocket stages that are to be used during orbital maneuver procedures, such as the Hohmann transfer or missions where reaching escape velocities is needed.

Research Question

How does an increase in rocket I_{sp} affect the change in mechanical energy of the rocket between the initial and modified engines, due to engine thrust?

The aim of the experiment is to investigate the relationship between rocket engine efficiency and the change in total rocket energy induced by engine thrust in orbit around a celestial body.

Hypothesis

In this experiment, the relationship between rocket I_{sp} and the change in mechanical energy induced by the engine is expected to be proportional because as the engine is more efficient, the work done by the said engine is expected to be relatively greater, provided that the engine has the same amount of propellant and an identical consumption rate.

Independent Variable

- Rocket Engine Specific Impulse (I_{sp})

The impulse responsible for the change in mechanical energy is delivered accordingly to the I_{sp} value of the engine. The higher this value, more work is done by the engine per unit mass of propellant ejected, changing the total energy of the rocket. Specific impulse will be changed by using different rocket engines. The effect of changes in I_{sp} values on ΔE_m will give out the relation being investigated in this experiment.

Dependent Variable

- Change in Mechanical Energy of the Rocket (ΔE_m)

The change in mechanical energy will be determined by calculating the difference between the initial and final mechanical energies. These energies will be calculated by using data collected pre- and post-burn per engine.

Controlled Varibles

- Initial Rocket Mass (m_i)

Since mass is a component of energy equations, mass throughout rockets should be stabilized. The difference in total mass of the rocket due to the utilization of different engines with various masses, will be compensated by adding deadweight to lighter rockets to keep the total mass constant.

- Mass Flow Rate ($\dot{m}$)

Mass flow rate of the propellant determines the duration of the burn. Higher $\dot{m}$ means less burn time but a larger force of magnitude. Although this relation seems to keep impulse delivered at an equilibrium, higher $\dot{m}$ results in higher impulse and therefore more change in mechanical energy. The mass flow rate will be kept constant by designating the smallest $\dot{m}$ value among the engines as standard and adjusting the mass flow rates of the other engines to this specific value. This adjustment will be done by tweaking the thrust percentages of the engines.

- Propellant Mass (m_p)

Given that the entirety of the fuel is consumed, the propellant mass equals the change in total mass, thus having a direct effect on ΔE_m. Additionally, with constant mass flow rate, more propellant would induce more impulse by extending the duration of the burn and therefore cause more ΔE_m. To prevent inconsistency all experimental rockets have an equal amount of propellant fuel.

Procedure and Steps of Experiment

Material List

- Kerbal Space Program simulation video game with mods:
  - Real Solar System —(to be able to experiment around Earth instead of in-game fictional planets and tweak engine values to be more realistic)
  - FASA —(to add a variety of rockets with different I_{sp} values)
  - MechJeb 2.0 and Kerbal Engineer Redux —(to take measurements in-game)*
  - Hyperedit —(to place rockets at their initial orbit)
  - Procedural Parts —(to be able to tweak total and deadweight mass)

*** MechJeb and KER provide information as to the following in-game;**

- Mass of rocket
- Speed of rocket
- Altitude of rocket

Setup

The purpose of this setup is to make a rocket with the designated total and propellant masses, and placing it to its initial orbit with determined altitude.

Total Mass (kg)	$2.0455 \cdot 10^4 \pm 5$
Propellant Mass (kg)	$5.3460 \cdot 10^3 \pm 5$
Initial Orbit Altitude (m)	$2 \cdot 10^6 \pm 1 \cdot 10^2$

Table 1 *Set Values for Experiment*

1. Click the Vehicle Assembly Building (VAB) in-game and start a new rocket.

2. Place an RC-L01 Remote Guidance Unit.

3. Place a Procedural Nose Cone with curve, diameter and length set respectively to Round #1, 2.5 meters and 1.375 meters on top of RGU.

4. Place an Advanced Reaction Wheel Module (Large) and a Z-4K Rechargeable Battery Bank under the with this order.

5. Place an Procedural RCS Tank at the bottom with diameter and length 2.5 and 0.5 meters respectively with full capacity.

6. Place a Procedural Ore Tank below the previous part and set diameter to 2.5 meter with full capacity.

7. Place a full capacity Procedural Liquid Tank with tank type as mixed, diameter and length at 2.5 and 1.25 meters respectively.

8. Place the engine of preference at the bottom.

9. Radially attach one MechJeb 2 module.

10. Read the mass of the rocket provided on the bottom left of the screen and calculate the difference between this value and the desired value of 20,455 kilograms.

11. Adjust the length of the Procedural Ore Tank in order to reach the desired mass by compensating the mass difference calculated in step 10.

12. Once completed click the Launch button on the top right of the screen.

13. Click the Hyperedit button on the righthand side of the screen.

14. Write the desired initial altitude of the rocket and press Set.

15. Press the “T” key on your keyboard to turn on SAS and locate and click the prograde button at the bottom center of the screen to direct the rocket.

16. Open the Utilities in the MechJeb tab on the top right of the screen and set the thrust limit to the percentage at which the mass flow rate of the current engine is equal the other experimental engines. (The information as to the max. force of thrust per engine can be found in the VAB to be used in the specific impulse equation to calculate mass flow rate.)

17. Throttle to the maximum the percentage restriction set previously allows by pressing and holding “Shift”.

18. Repeat for other rockets each functioning in different I_{sp} values.

Method

The following procedure will be repeated for all rockets with three trials in order to be able to determine any error that may occur during the simulation;

1. Note the speed and mass of the rocket.

2. Turn on the rocket engine by pressing "Spacebar".

3. Once the burn ends, note the mass of the rocket.

4. Write down the speed and altitude of the rocket at the **apoapsis**.

5. Calculate the initial and final mechanical energies of the rocket with measured values.

6. Find the difference between these values and determine ΔE_m for the rocket.

7. Repeat the procedure with different I_{sp} engine rockets.

8. Sketch the graph of ΔE_m values against their respective I_{sp} values.

Data Collected From Relevant Experiments

Engines	I_{sp} **(s)** (±0.5)	**Apoapsis Speed (m/s)** (±0.01)	**Apoapsis Altitude (m)** (10^4) (±1)	**Initial Mass (kg)** (10^4) (±1)	**Final Mass (kg)** (10^4) (±1)
Thor	495.0	3,005.76	1.6942	2.0456	1.5110
Vikas	527.4	2,780.46	1.9121	2.0455	1.5109
Regor A	561.6	2,545.44	2.1808	2.0457	1.5111
Apollo F1	576.0	2,446.79	2.3092	2.0454	1.5108
Vesta	612.0	2,204.74	2.6732	2.0456	1.5110
Hound	630.0	2,084.78	2.8851	2.0457	1.5111
Apollo Nova	675.0	1,788.52	3.5310	2.0458	1.5112
Apollo J2	702.0	1,613.37	4.0250	2.0460	1.5114
Vulcain 2	776.2	1,141.09	6.1156	2.0457	1.5111
RS-25 Shuttle Eng.	814.1	925.16	7.7842	2.0454	1.5108

Table 2 *Data Collected From Relevant Experiments of Specific Impulse vs. Change in Mechanical Energy*

Uncertainties on **Table 2** are determined by the smallest digit of the measurement device. Since all the trials held give the identical results due to the experiment being held in a simulation with no variation margin in its procedures, trial results are not provided in the table. The small values of uncertainty are also due to digital measurement of simulated values being exact and without random error.

Data Analysis

The calculations shown are for data gathered for Apollo F1 engine with an I_{sp} value of 576 s. Initially, the pre-burn mechanical energy of the rocket will be calculated.

Initial Mass (kg)	**Pre-burn Orbital Altitude (m)**	**Pre-burn Orbital Speed (m/s)**
20,454 ±1	$2 \cdot 10^6 \pm 1 \cdot 10^2$	6,900.49 ±0,01

Table 3 *Pre-burn Data*

The calculation for the initial mechanical energy is as follows;

$(GM(\mu) = 3.986 \cdot 10^{14}\ m^3s^{-2} \quad R_{earth} = 6.371 \cdot 10^6 \pm 1 \cdot 10^2\ m)$ (6)

$$E_i = \frac{-\mu \cdot m_i}{2R_i} = \frac{-(3.986 \cdot 20{,}454 \cdot 10^{14})\ (\pm 0.00489\%)}{2 \cdot (2 \cdot 10^6 + 6.371 \cdot 10^6)\ (\pm 0.00239\%)}$$

$$= \frac{-8.153 \cdot 10^{18}\ (\pm 0.00489\%)}{1.6742 \cdot 10^7\ (\pm 0.00239\%)} = -4.8698 \cdot 10^{11}\ (\pm 0.00728\%)\ kg\ m^2s^{-2}\ (J)$$

This value is the mechanical energy of the rocket prior to its orbital maneuver burn in a perfectly circular orbit around Earth. After the burn is completed the impulse delivered by the engine changes this energy value. In order to determine the investigated difference, the post-burn mechanical energy will be calculated. Unlike the calculation of the E_i, final mechanical energy will be calculated in two sections (E_k and E_g) due to the final orbit's elliptical shape.

The calculation for the kinetic component of the final mechanical energy at the apoapsis is as follows;

$$E_k = \frac{m_f \cdot v^2}{2} = \frac{15{,}108 \cdot 2{,}446.79^2 \ (\pm 0.00744\%)}{2}$$

$$= 4.5224 \cdot 10^{10} \ (\pm 0.00744\%) \ kg \ \ m^2 s^{-2} \ (J)$$

The calculation for the gravitational potential component of the final mechanical energy at the apoapsis is as follows;

$$E_g = \frac{-\mu \cdot m_f}{R_f} = \frac{-(3.986 \cdot 15{,}108 \cdot 10^{14}) \ (\pm 0.00662\%)}{(6.371 \cdot 10^6 + 2.3092 \cdot 10^7) \ (\pm 0.00343\%)}$$

$$= -2.0439 \cdot 10^{11} \ (\pm 0.00696\%) \ kg \ \ m^2 s^{-2} \ (J)$$

With both components of E_f of the rocket known, it is calculated as follows;

$$E_f = (4.5224 \cdot 10^{10} \pm 0.00744\%) + (-2.0439 \cdot 10^{11} \pm 0.00696\%)$$

$$= -1.5917 \cdot 10^{11} \ (\pm 0.0110\%) \ kg \ \ m^2 s^{-2} \ (J)$$

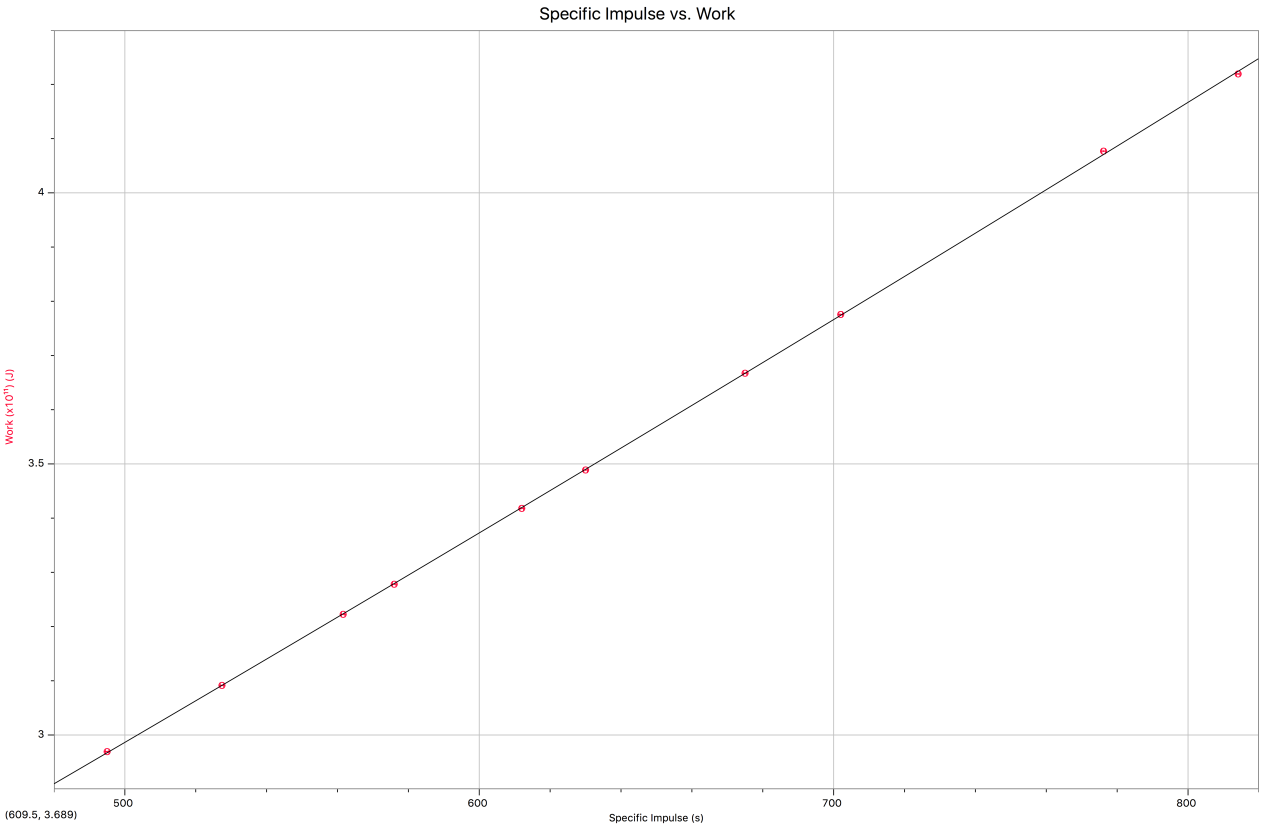
With both the E_f and E_i values obtained, the investigated difference in mechanical energy due to engine thrust is calculated as follows;

$$\Delta E_m = |E_f - E_i| = |(-1.5917 \cdot 10^{11}) - (-4.8698 \cdot 10^{11})|$$

$$= 3.2781 \cdot 10^{11} \ (\pm 0.00639\%) \ kg \ \ m^2 s^{-2} \ (J)$$

Engines	I_{sp} **(s) (±0.5)**	**Change in Mechanical Energy/ ΔE_m (*J*) (10^{11}) (±0.00639%)**
Thor	495.0	2.9693
Vikas	527.4	3.0915
Regor A	561.6	3.2226
Apollo F1	576.0	3.2781
Vesta	612.0	3.4180
Hound	630.0	3.4888
Apollo Nova	675.0	3.6673
Apollo J2	702.0	3.7757
Vulcain 2	776.2	4.0769
RS-25 Shuttle Eng.	814.1	4.2193

Table 4 *Calculated Change in Mechanical Energy Values with Corresponding Specific Impulse Variable Values*



Graph 1 *Specific Impulse vs Change in Mechanical Energy with Experimental Values*

Because of the irregular and narrow set of I_{sp} values due to the limitations of the simulation as to engine variety, determination of a relation graph requires non-experimental mathematical values. Firstly, the y-intercept of the graph should be known, the value of which correlates with ΔE_m induced by a zero-I_{sp} engine, as the x value on the graph is 0. This engine will have the same mass as the Apollo F1 rocket, thus the same initial mechanical energy of $-4.8705 \cdot 10^{11}\ J$. Since there is no orbital modification and only a change in mass, the final mechanical energy can be calculated with the mechanical energy equation for circular orbits, which is as follows;

$$E_{f_0} = \frac{-\mu \cdot m_f}{2R} = \frac{-(3.986 \cdot 15{,}108 \cdot 10^{14})\ (\pm 0.00662\%)}{2 \cdot (2 \cdot 10^6 + 6.371 \cdot 10^6)\ (\pm 0.00239\%)}$$

$$= -3.5970 \cdot 10^{11}\ (\pm 0.00901\%)\ J$$

$$\Delta E_{m_0} = |(-3.5970 \cdot 10^{11}) - (-4.8698 \cdot 10^{11})|$$

$$= 1.2728 \cdot 10^{11}\ (\pm 0.0533\%)\ J$$

Another important point on the graph is the I_{sp} value required to reach escape velocity with the given mass (Apollo F1), propellant and orbit of the rocket. The escape velocity is calculated as follows;

$$v_e = \sqrt{\frac{2\mu}{R}} = 9{,}758.77\ (\pm 0.00157\%)\ ms^{-1}$$

The change in velocity induced by the engine impulse to reach escape velocity is as follows;

$$\Delta v = |v_e - v_i| = |9{,}758.77 - 6{,}900.49| = 2{,}858.28\ (\pm 0.00571\%)\ ms^{-1}$$

This value is then put into the Delta-v equation to determine the I_{sp} value, as follows;

$$\Delta v = ln\left(\frac{m_i}{m_f}\right) \cdot I_{sp} \cdot g_0 \Rightarrow \frac{\Delta v}{ln\left(\frac{m_i}{m_f}\right) \cdot g_0} = \frac{2{,}858.28}{ln\left(\frac{20{,}454}{15{,}108}\right) \cdot 9.82} = I_{sp_e} = 960.76\ (\pm 0.0172\%)\ s$$

The final mechanical energy of a rocket with this I_{sp} value would be zero at infinity, therefore ΔE_m equals initial mechanical energy;

$$\Delta E_m = |E_i| = 4.8698 \cdot 10^{11} \ (\pm 0.00728\%) \ J$$

Any I_{sp} value beyond escape I_{sp} would result in HEV as the impulse delivered by the engine induces more change in velocity than required for escape. These rockets would have no E_g but would end up with E_k due to HEV, thus can be placed on the relation graph.

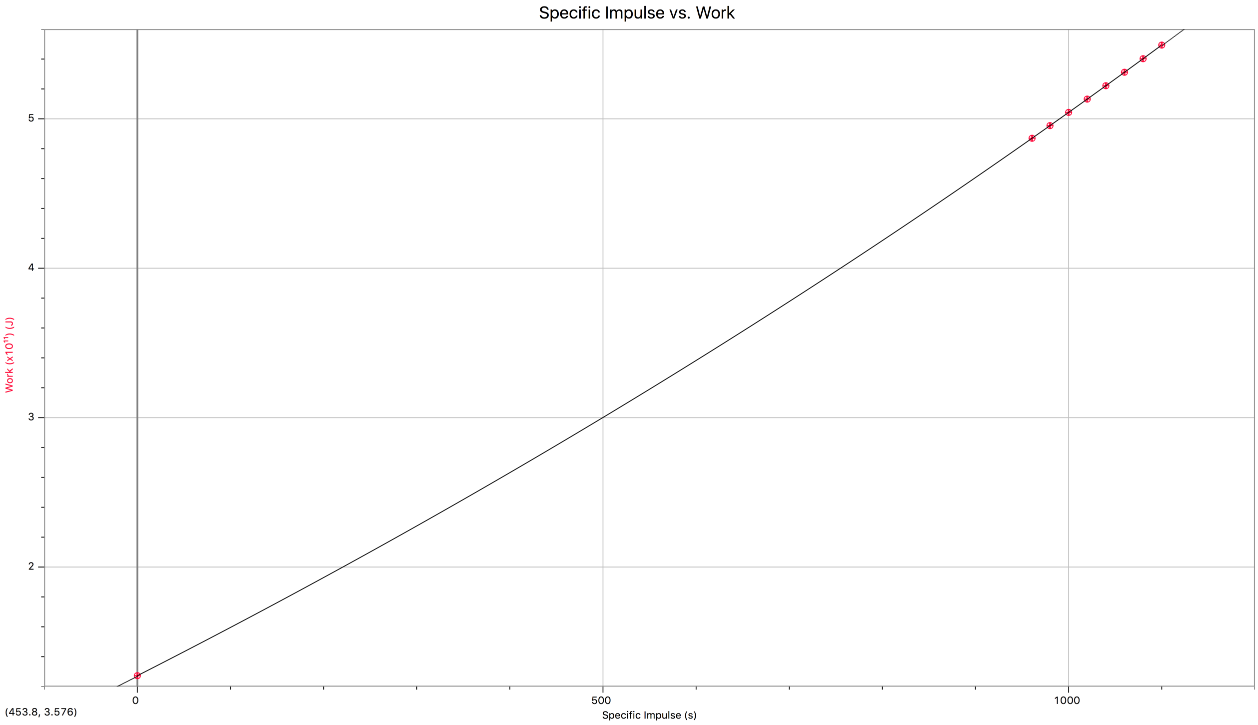
After the change in velocity due to an engine with an I_{sp} value greater than escape I_{sp} —980, for instance— is calculated via the Delta-v equation, HEV is calculated as follows;

$$v_\infty = \sqrt{\frac{-\mu}{a}} \Rightarrow \sqrt{\frac{-\mu}{\frac{1}{\left(\frac{2}{R}\right) - \left(\frac{v^2}{\mu}\right)}}} = \sqrt{\frac{-3.986 \cdot 10^{14}}{\frac{1}{\left(\frac{2}{8.371 \cdot 10^6}\right) - \left(\frac{9{,}816.00^2}{3.986 \cdot 10^{14}}\right)}}} = 1.0584 \cdot 10^3 \ (\pm 1.581\%) \ ms^{-1}$$

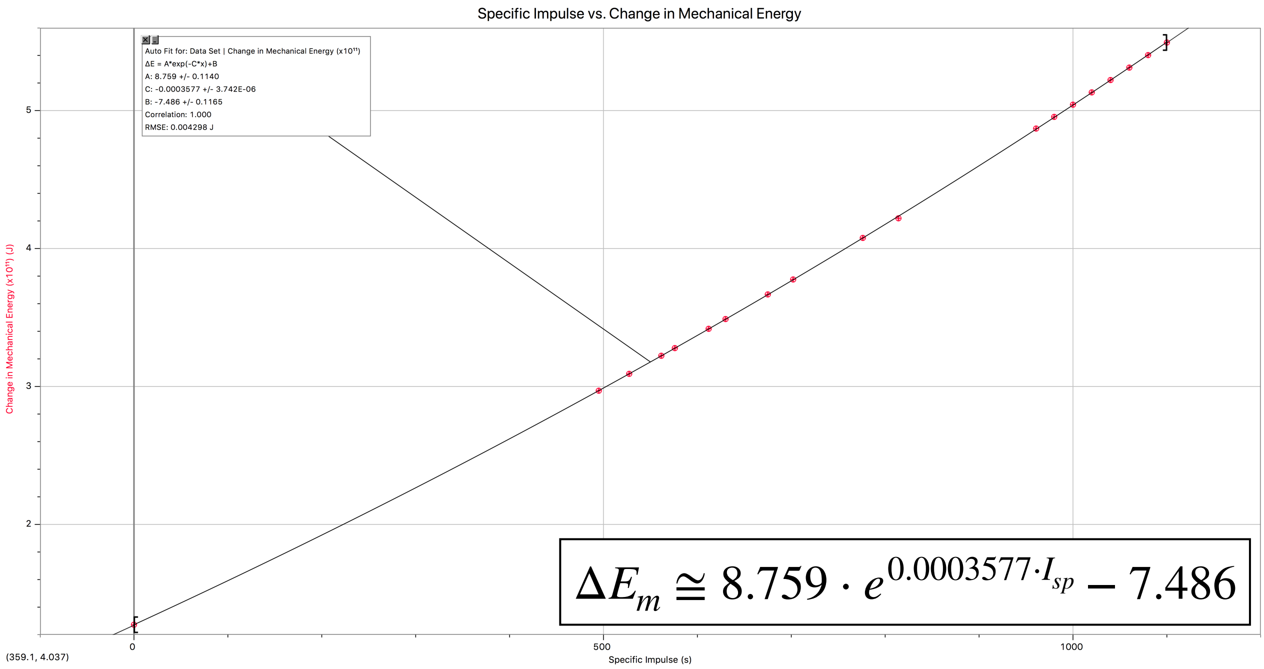
which is then used to calculate ΔE_m, which is determined to be $4.9544 \cdot 10^{11}$ J, by adding hyperbolic excess kinetic energy to initial E_m since all initial mechanical energy is nullified. Some HEV values calculated to be placed on the graph are listed in the table below.

I_{sp} **(s)** (± 0.5)	**Change in Mechanical Energy/** ΔE_m **(*J*)** **(10^{11})** ($\pm 0.0410\,\%$)
980.0	4.9544
1000.0	5.0429
1020.0	5.1320
1040.0	5.2215
1060.0	5.3117
1080.0	5.4023
1100.0	5.4935

Table 5 *Calculated Mechanical Energy for Hyperbolic Excess Velocity*



Graph 2 *Specific Impulse vs Change in Mechanical Energy with Mathematical Values*



Graph 3 *Specific Impulse vs Change in Mechanical Energy*

The graph above is drawn by including both experimental and mathematical data, where the set of values in the middle are simulated while the y-intercept and the data with greater values were mathematically calculated.

Conclusion and Evaluation

The purpose of this essay is to explore the relation between specific impulse of an engine and the change it induces on mechanical energy of the rocket. For each experiment, rockets with different I_{sp} engines, and identical total and propellant masses were used.

As visible from **Graph 3**, the relation is exponential ($\Delta E_m = A \cdot e^{-C \cdot I_{sp}} + B$) with percentage error of approximately 8.5 %, where lowest possible value is the y-intercept, zero-I_{sp}. Any value corresponding to negative I_{sp} suggests that either mass flow rate or force of thrust is negative in value, which is illogical. The reason why y-intercept is not at the origin is because the change in mass due to $\dot{m}$ causes a change in E_m.

Because of the narrow experimental set due to simulation constraints, without the addition of the y-intercept/zero-I_{sp}, the graph's curve would be imperceptible. The escape velocity and HEV values further support the exactitude of the graph drawn, indicating any value found from the graph is correct independent from whether the values can be experimentally tested, as values beyond escape I_{sp} cannot. With these values, it is apparent that the graph is not straight but an exponential curve. This is a sensible result due to the fact that during burn, the loss of mass reduces the force required to move the rocket, therefore resulting in more change in E_m.

Graph 3 is unique to total and propellant masses, and initial orbit of the rocket, in addition to the body orbited, Earth. However, in this essay it has been experimentally proven that the graph for different circumstances can be drawn by mathematically calculated values as it is an exponential graph satisfying three mathematical values (zero-I_{sp}, escape I_{sp} and one beyond escape I_{sp}). The graph can be modified to any rocket with known total and propellant masses and starting orbit, provided it orbits Earth. From this graph, the range of possible orbit modifications for the rocket with known I_{sp} is calculable, since the graph can be read as the corresponding ΔE_m values are the maximum change possibly induced by the I_{sp} value.

Because the simulation relies on realistic physical properties at small scales, only losing reliability at cosmic scales, which is of no concern in this essay, experimental values are coherent with mathematical calculations. Since experiments are held in a computer simulation, random error is almost negligible, the only potential cause for which would be the positioning of immediate celestial bodies in-game, such as the Moon. However, because the experiment is held in Low Earth Orbit, such an effect would be inconsequential.

To summarize, the relation between specific impulse of rocket engines is exponential with the y-intercept being the lowest possible value of a hypothetical zero-I_{sp} engine. Since the relation is now known, this graph can be drawn for any rocket with required information of certain properties obtained, with applicability in the real world. The hypothesis is correct for this experiment, as the values of ΔE_m increase with the values of I_{sp}.

To further the research, effects of controlled variables on ΔE_m value may be explored. The consideration of how these factors modify the relation would help the determination of a standardized equation that is applicable to any rocket under any circumstance, removing the assumptions of all the propellant being consumed etc. After the equation is determined, consumption efficiency at different points on an elliptical orbit for a given modification can be explored for not only prograde/retrograde burns, but also radial in/out. It should be taken into account that E_m values show distinct change only when the modification is on the same orbital plane as the initial orbit. Therefore, inclination burns (normal/anti-normal) should be approached with a different manner than suggested in this investigation.

References

(1) Benson, Tom (2015, October 22), *Ideal/Tsiolkovsky Rocket Equation*. Retrieved from https://spaceflightsystems.grc.nasa.gov/education/rocket/rktpow.html

(2) Hall, Nancy (2015, May 5), *Specific Impulse*. Retrieved from https://www.grc.nasa.gov/www/k-12/airplane/specimp.html

(3) *Hohmann Transfer Orbits*. Retrieved from http://burro.astr.cwru.edu/Academics/Astr221/Gravity/transfer.html

(4) Tsokos, K. A. (2014). Circular Motion and Gravitation. *Physics for the IB Diploma* (pp. 249-265). Cambridge, United Kingdom: Cambridge University Press.

(5) Braeunig, Robert A. (2013), *Orbital Mechanics*. Retrieved from http://www.braeunig.us/space/orbmech.htm

(6) Stengel, Robert (2016), *Orbital Mechanics*. Princeton University. Retrieved from http://www.princeton.edu/~stengel/MAE342Lecture2.pdf