TED ANKARA COLLEGE FOUNDATION PRIVATE HIGH SCHOOL

A FIELD TRIP IN A NON-EUCLIDIAN GEOMETRY

Mathematics Extended Essay

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D1129-0049

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"The process of setting up a mathematical model of a real situation nearly always involves making simplifying assumptions." Eugene F. Krause

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Abstract

From elementary school to high school we have been learning different types of geometry. Taxicab geometry is an example of non-euclidean geometry. Although it has similar points with Euclidian geometry, it differs from Euclidian geometry in an area; in distance functions. I have decided to try a new geometry. So, I have found taxicab geometry. When I researched about taxicab geometry, I have asked myself that "Could I turn this geometry into three dimensions?"

My research subject is "Taxicab Geometry in Three Dimensions". In the first part of the essay, my reasons of choosing this subject and an explanation of taxicab geometry are briefly given. After this quick information in the second part I will be proving Taxicab geometry is a metric*. Third part involves Euclidian geometry's theorems for distance and theorems that are highly related to distance functions in three dimensions are given. Also formulas for distance in two dimensional taxicab geometry are given a. Furthermore, these theorems are taken as reference points for converting taxicab geometry to three dimensions. Distance function in two dimensions is converted to third dimension in this part. In the fourth part, sphere and cylinder which belongs to Euclidian geometry are recreated by me in taxicab geometry using the formulas that are found in the third part. The last part of the essay is about applications of taxicab geometry "Taxicab Geometry in Three Dimensions". The taxicab space's part in daily life, city planning and in buildings is explained by an example. To conclude, I will try to find answer to the question "Is it possible to convert sphere and cylinder from three dimensional Euclidian geometry to three dimensional taxicab geometry?"

WORD COUNT: 292

**A metric is a mathematical function that measures distance (Christina Janssen, July 2007, page 5)*

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1.Introduction

1.1 Reasons of Choosing Taxicab Geometry

I have always been interested in unusual and unordinary things. I have got the same interest in science and mathematics too. So, I have always searched for theorems and systems that are not being known by society. One of these researches brought me to "Taxicab Geometry". When I have looked for more, I have seen that "Taxicab Geometry" is being used in urban planning an even in games(SimCity). Therefore, as one of my dream job is civil engineering, I have immediately chosen this subject as I heard about extended essays.

1.2 What is Taxicab Geometry?

"The usual way to describe a (plane) geometry is to tell what its points are, what its lines are, how distance is measured, and how angle measure is determined." (Krause E. F. 1975 page 2). In taxicab geometry, lines and points are the same as they are in Euclidian geometry. So, we can define point as a location; line as points aligned together and has no thickness or a width. In addition, the amount of turn between two straight lines that have a common end point will be the definition of angle. The difference between Euclidian Geometry and Taxicab Geometry is the distance functions/formulas.

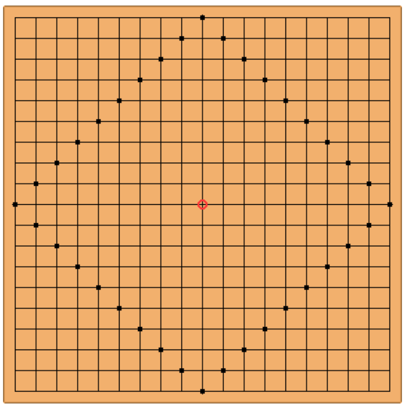


Figure 1.2: Taxicab Coordinate system

In Euclidian Geometry for points A(x_a , y_a) and B(x_b, y_b) d_E(A,B)=$\sqrt{(x_a - x_b)^2 + (y_a - y_b)^2}$ gives the distance between points A(x_a , y_a) and B(x_b, y_b). However Taxicab Geometry takes d_T(A,B) = $|x_a - x_b| + |y_a - y_b|$ as the distance function between points A(x_a , y_a) and B(x_b, y_b).

2. The Proof of Taxicab Geometry Is a Metric

2.1 What is a metric?

Metric is the function that measures distance between points. In order to measure the distance between points, a geometry system has to fulfill the given three axioms of metric:

"Let P, Q, and R be points, and let d(P,Q) denote the distance from P to Q.

Metric Axiom 1 d(P,Q)≥0 and d(P,Q) = 0 if and only if P= Q

Metric Axiom 2 d(P,Q) = d(Q,P)

Metric Axiom 3 d(P,Q) +d(Q,R)≥d(P,R) " (*Christina Janssen, July 2007,page 5)*

<u>So let A(x_a , y_a) , B(x_b, y_b) and C(x_c,y_c) our points and they are ∈ $\mathbb{R}^2$</u>

For Axiom 1:

- If A≠B d(A,B)>0 and if A=B d(A,B)=0.
- If A≠B then either x_a≠ x_b or y_a≠ y_b
- Therefore either <u>only one</u> of this will happen $|x_a - x_b| > 0$ $|y_a - y_b| > 0$ so d(A,B)>0

- If A=B then $x_a = x_b$ or $y_a=y_b$ Therefore $|x_a - x_a| = 0 \; and \; |y_a - y_a| = 0$ so d(A,B)=0
- So Taxicab fulfills the first axiom

For Axiom 2:

- Taxicab geometry should fulfill the statement d(A,B)=d(B,A)
- So $|x_a - x_b| + |y_a - y_b|$ must be equal to $|x_b - x_a| + |y_b - y_a|$
- $|(-1)(-x_a + x_b)| + |(-1)(-y_a + y_b)|$
- $|(-1)(x_b - x_a)| + |(-1)(y_b - y_a)|$
- $|(-1)| \; |(x_b - x_a)| + |(-1)| \; |(y_b - y_a)|$
- Therefore $|x_b - x_a| + |y_b - y_a|$
- To conclude Taxicab Geometry fulfills the Axiom 2 of metric system too

For Axiom 3:

- d(A,B)+d(B,C)≥d(A,C)
- Let A(x_a , y_a) , B(x_b, y_b) and C(x_c,y_c) be are non-collinear and forming a triangle
- d(A,B)+d(B,C)= $|x_a - x_b| + |y_a - y_b| + \; |x_b - x_c| + |y_b - y_c|$
- $(|x_a - x_b| + |x_b - x_c|) + (|y_a - y_b| + |y_b - y_c|)$
- $(|x_a - x_b \; + \; x_b - x_c|) + (|y_a - y_b + y_b - y_c|)$
- $|x_a - x_c| + |y_a - y_c|$
- $= d(A, C)$

To sum up, Taxicab Geometry fulfills all of the axioms of metric. Therefore, it is a metric.

3. A Baby Step To Three Dimensions

3.1 Distance Functions of Euclidian Geometry in Three Dimensions

In the part 1.2 the distance functions for Euclidian Geometry and Taxicab Geometry are given. However, let's see them again; for Euclidian Geometry for points A(x_a , y_a) and B(x_b, y_b) d_E(A,B)=$\sqrt{(x_a - x_b)^2 + (y_a - y_b)^2}$, for Taxicab Geometry it is d_T(A,B) = $|x_a - x_b| + |y_a - y_b|$.

So the question is "How does distance functions in two dimensional Euclidian Geometry is converted to three dimensions?". As the logic and system when converting two dimensional Euclidian Geometry to three dimensions is taken as a reference in this essay, the question needed to be answered.

Only difference when converting two dimensions into three dimensions is; there is an extra coordinate axis which is called as "z". Therefore, the distance between the points on "z" axis needed to be calculated too. So the distance formula between points A(x_1, y_1, z_1) and B(x_2, y_2, z_2) in 3-dimensions is shown at Figure 3.1.

$$d_E(A,B) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

Figure 3.1

As the distance function in 2-dimensional taxicab geometry is given at the chapter 1.2, it has to be calculated the distance between the points "z" coordinates too. It is known that taxicab distance is the shortest path between two points. Therefore, the distance between "z" coordinate of points is their differences absolute value.

3.2 Converting To Three Dimensions

Since the distance function of taxicab geometry in 2-dimensions is d_T(A,B) = $|x_a - x_b| + |y_a - y_b|$;

3-dimensions function will be $d_T(A,B) = |x_a - x_b| + |y_a - y_b| + |z_a - z_b|$ for the reasons in chapter 3.1.

4. New Type Of Geometric Shapes

4.1 About Cylinder and Sphere

Sphere is defined as the perfectly round geometrical shape in Euclidian Geometry in 3-dimension. Like a circle in two dimensions it has a center which is at equal distance to all of the points on the sphere's surface. In fact, the distance between center and these points is called radius or shortly *r*.

Cylinder is formed when a rectangle is rolled around itself and forms two circular bases. (Figure 4.1)

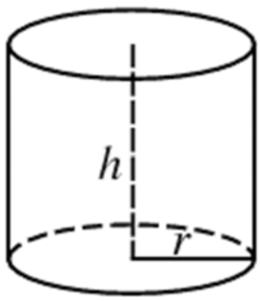


Figure 4.1

As it can be understood both of these solid figures which belongs to Euclidian Geometry has a relation with circles. First step before converting these to 3-dimensions a circle in taxicab geometry has to be explained.

In Euclidian Geometry, circle is defined as: the locus of points on a plane that are at the equal distance to a single point which is called as center. As taxicab geometry does not have any "roundness" and It is based on squares, circle should be a square type shape, thus every point on that shape has the same distance to the center. Figure 4.2 shows a circle in taxicab geometry with a radius of 3 units.

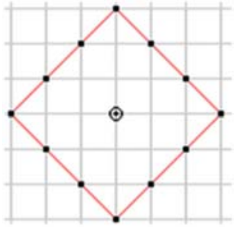


Figure 4.2 / At http://www.mathematische-basteleien.de/taxicabgeometry.htm

the equation for a circle in taxicab geometry with the center C (0, 0) is given as $|x|+|y|=r$.

As a circle is defined as *the locus of points on a plane that are at the equal distance to a single point which is called as center*, we can enlarge this equation by giving the equation of a circle in taxicab geometry which has the center C(x_0, y_0) is $|x - x_0| + |y - y_0| = r$.

4.2 Converting To Taxicab Sphere

As it is done at the chapter 3, all of the points should be at the same distance from the center, including the "z" axis to.

Finding the equation: Let's take a center C(x_0, y_0, z_0) where it is $\in \mathbb{R}^3$

- It has to be at equal distance to all of the points on the sphere and the distance is called radius
- As the distance formula for 3-dimensional taxicab geometry is $d_T(A,B) = |x_a - x_b| + |y_a - y_b| + |z_a - z_b|$ then the distance between center and other points should include this and should be equal to the radius.
- Therefore $|x_c - x_0| + |y_c - y_0| + |z_c - z_0|$= r and this is the equation for sphere in 3-dimensions.

There are a lot of arguments about the shape of sphere in taxicab geometry. Christina Janssen in 2007, suggested that it should be look like two cones stick together from their bases. However, this shape does not satisfy the taxicab's squarish geometry and also it does not fulfill the condition for a sphere which is " all points are at equal distance from the center". So, Kevin P. Thompson used two right triangles sticked together from their square bases to each other to describe a sphere.

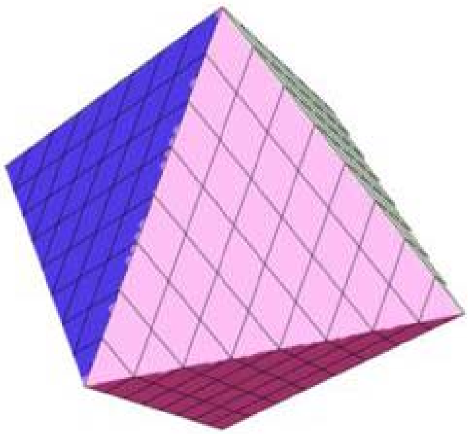


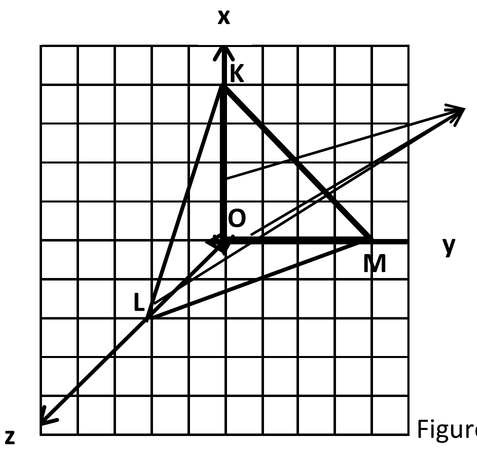
Figure 4.3: A sphere in Taxicab Geometry taken from

[1] http://inspirehep.net/record/916286/plots

If there is a solid figure than this figure has a volume and a surface area too.

4.3 Equations for Taxicab Sphere's Volume and Area

As a sphere in taxicab geometry is two right square pyramids sticked up together the volume and surface area equations could be found from it.



The arrows shows the radius

Figure 4.4/ this figure shows the one side of a sphere in taxicab geometry

In Figure 4.4 one side of a sphere is given. To calculate the area of one side:

- |OK|=|OM|=|OL|=r
- |OK |$\perp$|OL|$\perp$|OM|
- From the Pythagorean theorem which is "$a^2+b^2=c^2$, r^2+r^2=|LM|^2=|KL|^2=|KM|^2
- $2r^2$=|LM|^2=|KL|^2=|KM|^2
- $r\sqrt{2}$=|LM|=|KL|=|KM|
- The height of the triangle KLM is equal to $\frac{\sqrt{3}}{2}$ x $r\sqrt{2}$ since KLM is an equatorial triangle. Therefore the height will be r$\frac{\sqrt{6}}{2}$
- The area of a triangle is the product of its height and base area divided by two.*

*http://www.math.com/tables/geometry/areas.htm

- So, the area of KLM is ($r^2\sqrt{3}$)/2

- Since there are 8 of these triangles on the sphere, then the surface area equation for a taxicab sphere will be = 4 $r^2\sqrt{3}$

For Volume:

- Since there are two right square pyramids it is needed to calculate the two pyramids volume
- A right square pyramids equation for volume is: $\frac{1}{3}(h.b^2)$ where h is the height and b is the base side length*(http://math.about.com/od/formulas/ss/surfaceareavol_5.htm)
- Thus, for a taxicab square the volume equation will be : 2x($\frac{1}{3}2r^3$)

4.4 Cylinder in Taxicab Space

In taxicab space as it was mentioned in chapter 4.1, a circle is an equilateral quadrilateral. Therefore, a cylinder's bases in taxicab space should consist of two equilateral quadrilaterals. Figure 4.5 shows a cylinder in taxicab space.

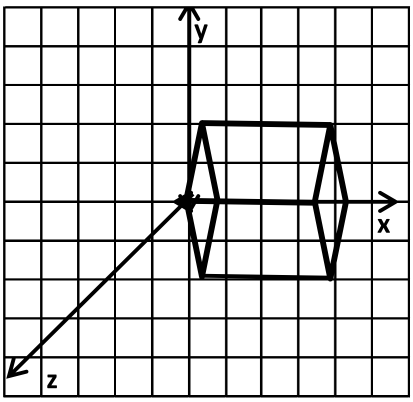
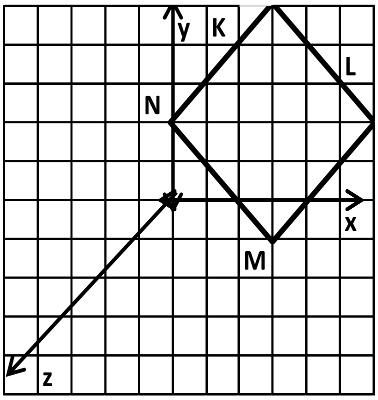


Figure 4.5

To calculate the cylinders volume, the base area and height must be known. For the base area of the cylinder, the area equation for circle in taxicab space should be known.



How to calculate a circle's area in taxicab space:

- Let's take the circle KLMN in taxicab space
- As it is said to be a square in taxicab space the area should be two sides product however when we use this technique in taxicab space it does not work because when we measure the length of KL in taxicab metric the result will be 6 units. It is not reasonable to say that the area of what appears as a square has area equal to the side squared in taxicab space because if it was so, the area of KLMN will be 36 units however if the squares in it counted, the result will be 18.
- There are two isosceles right triangles in KLMN
- The sides for these triangles is r. Because of this, the area of a triangle is $\frac{r^2}{2}$. Thus, the area of the circle will be $2r^2$.

To sum up the volume of a cylinder in taxicab space will be = h. $2r^2$ where h is the height of the cylinder and r is the radius.

5. Applications of Taxicab Space

5.1 The Toilet Problem

Taxicab geometry is the perfect geometry to use in urban geography. Euclidian geometry measures where the "crow flies". However in cities cars cannot fly and taxicab geometry provides them to calculate the distance between two points and how much do they have to travel. It is used in urban planning. "The Toilet Problem" is a problem that I have found where taxicab space could be used. It can be used to calculate the distance that has to be travelled up, down and forward/backwards. The problem goes with:

"In an Ideal city, a square prism building KLMOSRPN with the corners K(0, 0, 4), L(4,0,4), M(4,0,0), O(0,0,0), S(0,8,0), R(0,8,4), P(4,8,4) and N(4,8,0) there are offices in every corner. The workers want a bathroom in the building. The engineer decides to build this bathroom where everybody walks the same distance. Where should he/she build it? (We assume that this building exist in taxicab space)"

In this question the equation that has been written in the chapter 3.2 will be used.

To start with let's take the bathroom's coordinate as B(x, y, z) $\in \mathbb{R}^3$

- d_T(K,B) = $|0-x|+|0-y|+|4-z|$= x + y + |4-z| as B(x, y, z) is in the building x, y and z can't be a negative number and can't be bigger than 4 so d_T(K,B)= x+y+4-z
- d_T(L,B) = $|4-x|+|0-y|+|4-z|$= (4 – x) + y + 4 – z= 8 – x + y –z
- d_T(M,B) = $|4-x|+|0-y|+|0-z|$ = (4 – x)+y + z
- d_T(O,B) = $|0-x|+|0-y|+|0-z|$= x + y + z
- d_T(S,B) = $|0-x|+|8-y|+|0-z|$= x + (8 – y) + z
- d_T(R,B) = $|0-x|+|8-y|+|4-z|$= x + (8 – y)+ z
- d_T(P,B) = $|4-x|+|8-y|+|4-z| = 4-x+8-y+4-z$

- $d_T(N,B) = |4 - x| + |8 - y| + |0 - z| = (4 - x) + (8 - y) + z$
- All of the equations above should be equal to each other. Just solving the first two step's equation will get us: 8 – x + y –z= x+y+4-z when we add –y and +z to both sides we will have 8-x=4 + x, adding -4 to both sides will lead us to 4-x=x adding +x to both sides will show us the final equation of this system which is 4=2x. Dividing both sides to 2 will give us the x which is x=2. You can test the answer by putting it instead of variable x and testing if it satisfies or not.
- The 5^{th} and 4^{th} step will give us the y. x + y + z= x + 8 – y + z, adding –x and –z to both sides will turn the equation to y=8-y and adding +y to both sides; 2y=8. Now we can simplify the both sides of the equation and our variable is y=4. You can test the answer by putting it instead of variable y and testing if it satisfies or not.
- The 7^{th} and 8^{th} steps will give us the z. 4 - x + 8 - y + z = 4 - z + 8 – y + 4 – z. Adding (x+4)+(y+8)+ z to both sides of the equation will give us the final equation; 2z=4 and simplifying by 2 will give us the z=2. You can test the answer by putting it instead of variable z and testing if it satisfies or not.

So the bathroom should be built at B(2, 4, 2).

Another way to solve this is finding the mid-spot which is B($\frac{\sum_1^n x_n}{n}, \frac{\sum_1^n y_n}{n}, \frac{\sum_1^n z_n}{n}$)

6. Conclusion and Evaluation

In this essay I have tried to find an answer to the question “Is it possible to convert sphere and cylinder from three dimensional Euclidian geometry to three dimensional taxicab geometry?” I have studied to find an answer to this question. I have tried to convert it and succeed in this mission. I have converted a sphere into taxicab space and saw that the conception of sphere has changed completely in taxicab space. I have discovered the sharp edges of taxicab and I prefer them, rather than roundness of Euclidian Geometry in some areas. However there are some weaknesses of my study. These results are not a hundred percent officially true because this geometry is changing and new theorems are going on.

This subject is first proposed by Herman Minkowski in early 20s and then it is first published to public opinion in 1975 by Eugene Krauser who has written the book about how to use taxicab and its rules. On the upshot this is a very fresh subject. Therefore, there is a lot of arguing going on about this non-Euclidian geometry. In addition, there have been a lot of arguments about the shapes in taxicab space too. For example, Christina Janssen has tried to draw a sphere but it was an unsuccessful trial (she is the one who made me like this subject by reading her thesis). So the problem is this geometry’s popularity is growing exponentially and this leads to new arguments and changes. To be honest this was the reason why I have chosen this subject. Although, it has some disadvantages we are using it unconsciously in our daily life. For example the navigators, while we are playing chess and even at our computer games (Sim City). It has already got into our life’s, . In the whole world there are not much of professors who are deeply into this subject. However, I believe this geometry has future. It can help us to plan our cities and urban life’s better than any geometry type. Besides all of the excitement, fun and a new geometry type, this geometry is like living and anyone could feed this monster with their study.

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