MATHEMATICS EXTENDED ESSAY

“Relationship Between Mathematics and Art”

Candidate Name: Zeynep AYGAR

Candidate Number: 1129-018

Session: MAY 2013

Supervisor: Mehmet Emin ÖZER

TED Ankara College Foundation High School

ZEYNEP AYGAR
D1129018

ABSTRACT

For most of the people, mathematics is a school course with symbols and rules, which they suffer understanding and find difficult throughout their educational life, and even they find it useless in their daily life. Mathematics and arts, in general, are considered and put apart from each other. Mathematics represents truth, while art represents beauty. Mathematics has theories and proofs, but art depends on personal thought. Even though mathematics is a combination of symbols and rules, it has strong effects on daily life and art.

It is obvious that those people dealing with strict rules of positive sciences only but nothing else suffer from the lack of emotion. Art has no meaning for them. On the other hand, artists unaware of the rules controlling the whole universe are living in a utopic world. However, a man should be aware both of the real world and emotional world in order to become happy in his life. In order to achieve that he needs two things: mathematics and art. Mathematics is the reflection of the physical world in human mind. Art is a mirror of human spirit. While mathematics showing physical world, art reveals inner world. Then it is not wrong to say that they have strong relations since they both have great effects on human beings. In this study, the relation between art and mathematics are shown by the examples of, Golden Ratio, Fibonacci sequence, and music.

Talking about the relation between mathematics and arts, we frequently coincide some mathematical facts such as Golden Ratio and Fibonacci sequence. In fact, we see their applications on many items and even in nature. In this study, I will try to investigate the relation between mathematics and art and these concepts in detail.

Word Count: 289

CONTENTS

RESEARCH QUESTION

Are mathematics and art related?

1. INTRODUCTION

For some people, mathematics and arts are two different field of study which have no relations at all. In reality, this is not so. Mathematics has aesthetic, and art in itself and it has interrelation with the different fields or arts such as architecture, painting, poetry and music, and even in nature. While investigating the relation between mathematics and art, we frequently face with some mathematical concepts such as Golden Ratio, Fibonacci sequence, and Pythagoras Comma etc.

How do these concepts are related to arts? Before going further, I will briefly describe what these concepts are.

2. GOLDEN RATIO

2.1. Definition

The golden ratio, indicated by φ, is a geometric and numeric ratio of the parts of a whole that gives the most appropriate dimensions. It can be also called golden mean, divine proportion or golden section. Two objects are said to be in golden ratio if the sum of two objects to the larger one equals the ratio of the larger one to the smaller one. This can be simply explained on a line as explained in Figure 1 below:

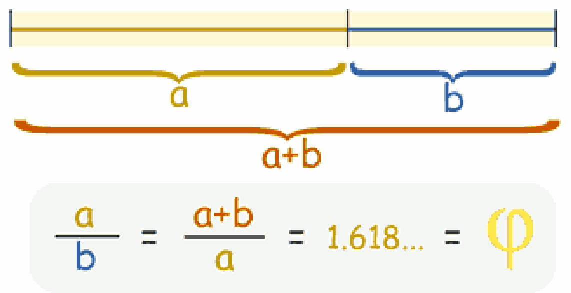


Figure 1: Golden Ratio on a line.

This explanation of golden ratio was first given by Euclid ca. 300 BC in an equivalent definition so-called "extreme and mean ratios". Euclid stated that the total length of a line divided by the longer part is equal to longer part divided by shorter part. Thus, a perfect division is generated. This ratio is written as

$$a^2 - ab - b^2 = 0 \qquad (1)$$

If we take b = 1 then we have

$$a^2 - a - 1 = 0 \qquad (2)$$

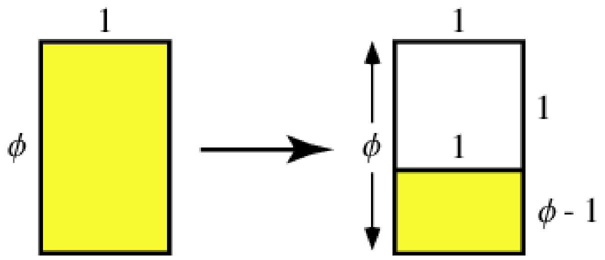
The solution of this quadratic equation gives that

$$a = (1 + \sqrt{5})/2 \qquad (3)$$

$$= 1.6180339887498948482045863436563811 7720 \ldots \qquad (4)$$

This number is called golden ratio and is indicated by the Greek letter φ. The reciprocal of golden ratio φ is called "phi" and for larger quantities of golden ratio is defined by "Phi". Phi has connections with continued fractions and Euclidian algorithm for computing the greatest common divisor of two integers.

When we use the same logic on a two dimensional objects such as a rectangle with dimensions of φ and 1, we get the rectangle shown on Figure 2(a).



(a) (b)

Figure 2 : Golden rectangle.

Now, if we draw a square with length 1 within the rectangle of Figure 2(a), we get a square of size 1 and a new rectangle within the original one as shown in Figure 2(b) above. Notice that the yellow colored rectangles of two sketching Figure 2(a) and 2(b) are similar. Then if we apply Euclid's rule in Figure 2(b), we get

$$\varphi/1 = 1/(\varphi-1) \tag{5}$$

Then we have

$$\varphi^2 - \varphi - 1 = 0 \tag{6}$$

Notice that equations (2) and (6) are the same. So for the quadratic equations in (6) we get the same solution as in equation (3) resulting

$$\varphi = (1 + \sqrt{5})/2 \tag{7}$$

$$= 1.6180339887498948482045863436563811772\,0 \ldots \tag{8}$$

The rectangle with the properties given above is called Golden Rectangle, which gives the most aesthetic view for the eye.

In order to obtain a golden rectangle, a square with side 1 is drawn and the base line is extended outside of the square. From the middle point of the base of the square, a semicircle with the radius √5/2 is drawn by use of a compass. Notice that this semicircle passes over the upper corners of the square. If we complete the rectangle from the point that semicircle cuts the extended base line, we obtain a perfect golden rectangle as shown in the Figure 3 below.

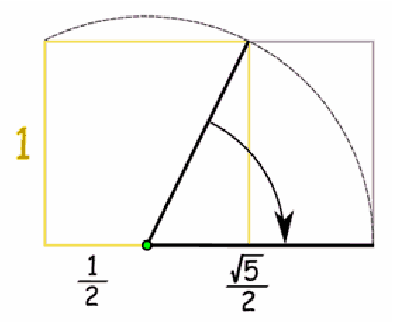


Figure 3 : Sketching golden rectangle.

As seen from the Figure3, if one side (shorter side) of the rectangle is 1 unit long, then the longer side is (1 + √5)/2 in length, as calculated from the quadratic equations given in the equations (2) and (6).

Successive points dividing a golden rectangle into squares and smaller rectangle as drawn on Figure 2(b) gives a logarithmic spiral that is mostly called the golden spiral, as shown on Figure 4 below.

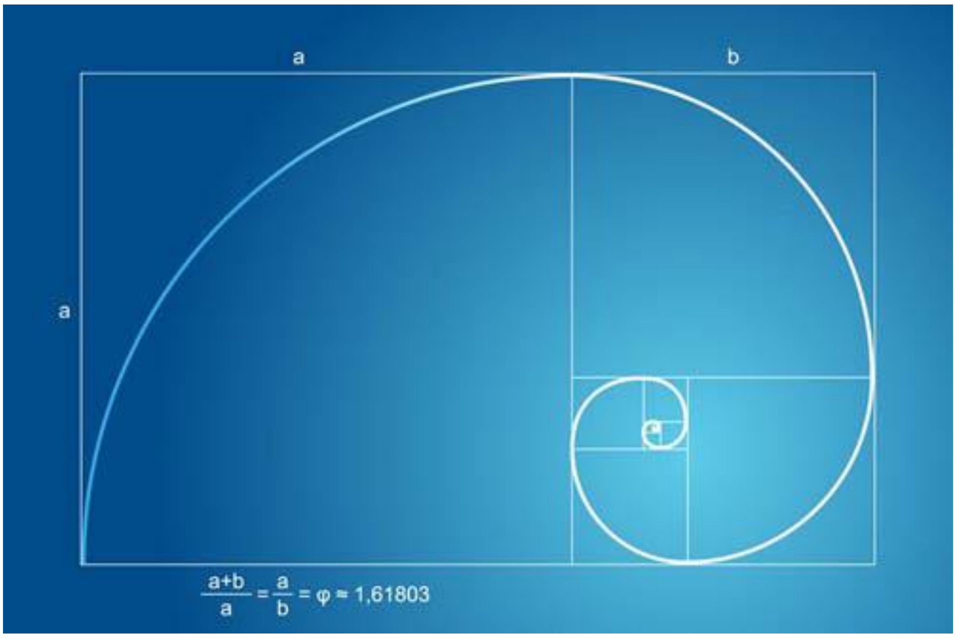


Figure 4 : Golden Spiral.

2.2. Applications

The great German mathematician and astronomer Johannes Kepler described the golden number as one of the “two great treasures of geometry.” (The other is the Theorem of Pythagoras.) Golden number appears in many basic geometric constructions and applications:

2.2.1. 3 lines

On a two dimensional space, if we take 3 equal lines and put one on Y axis, then put the 2nd line against the midpoint of the 1st line to X axis, and once more put the 3rd line against the midpoint of the 2nd. Line to X axis, we obtain the construction shown on Figure 5. In this figure, the ratio of AG to AB is Golden Number.

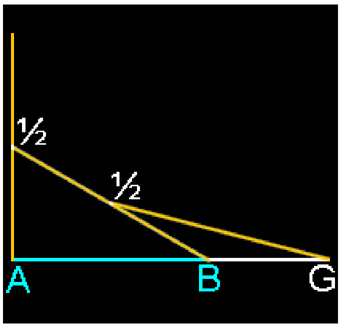


Figure 5 : Golden Ratio obtained by 3 equal lines.

2.2.2. 3 sides: Triangle

If we insert an equilateral triangle inside a circle and add a line at the midpoint of the two sides of the triangle and extend this line to the circle surrounding the triangle as in Figure 6, the ratio of line segment AG to line segment AB is Golden Number.

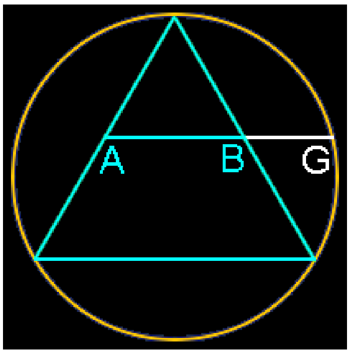


Figure 6 : Golden Ratio in a triangle within a circle.

2.2.3. 4 sides: Square

In a similar way, if we insert a square inside a semi-circle as in Figure 7, we find that the ratio of line segments AG to AB is Golden Number.

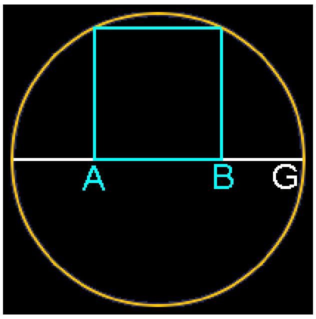


Figure 7 : Golden Ratio in a square within a circle.

2.2.4. Pyramid

In a square pyramid the apothem is the height of its lateral face. It is calculated by Pythagorean Theorem with the height of the center of the pyramid and half of its base length.

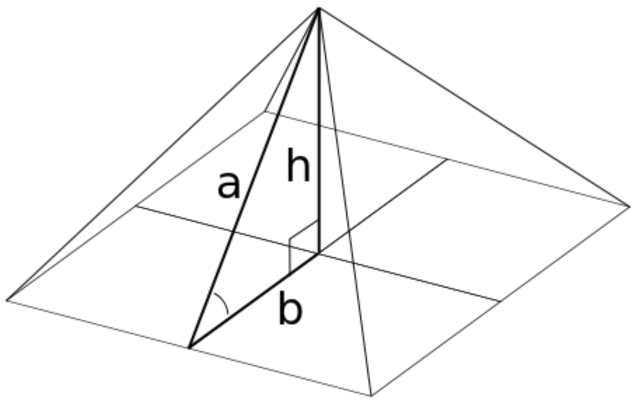


Figure 8 : Apothem of a Pyramid

Thus we get apothem (a in the Figure 8) as

$$a^2 = h^2 + b^2 \tag{9}$$

For a pyramid if a = φ x b, then this pyramid is called golden pyramid. Now, if we assume that b = 1 and a = φ, then we have

$$\Phi^2 = h^2 + 1 \quad \text{or} \quad \Phi^2 - h^2 - 1 = 0 \tag{10}$$

which has the same form as the equation (6). Comparing equations (6) and (10) we get

$$h^2 = \Phi \tag{11}$$

or

$$h = \Phi^{1/2} \tag{12}$$

Then we have a square pyramid with the apothem = Φ, height = $\Phi^{1/2}$ and semi base length =1 as shown in Figure 9 below.

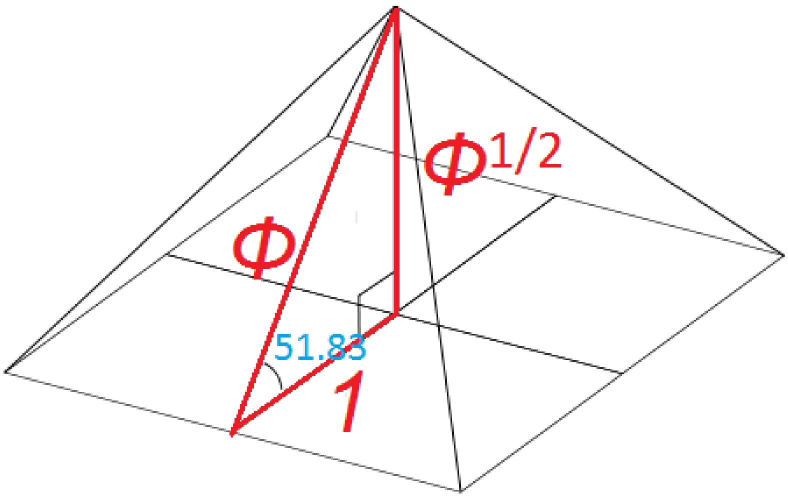


Figure 9 : Square pyramid with golden ratio.

The right triangle drawn by red lines within this pyramid will have a base angle of 51.83 degrees. Using trigonometric tables we get that *cos 51.83* = 0.61804697 which is equivalent to the reciprocal of golden ratio (0.61804697 =1/1.61803398874989484).

3. GOLDEN RATIO IN ARHITECTURE

3.1. Pyramids of Egypt

The Egyptian pyramids have very close constructions to the golden pyramid. Within those pyramids, one, the Great Pyramids of Giza (also known as Cheops) has a slope of 51.52 degrees which is very close to the slope of the golden pyramid which is 51.83 degrees. The other pyramids such as Chephren with a slope of 52.20 degrees and Mycerinus which has a slope of 50.47 degrees are very close to the golden pyramid also.

There are many other buildings of ancient world which golden ratio, golden rectangle or golden pyramid used widely in their constructions.

3.2. Parthenon in Athens

The Parthenon in Athens, built by the ancient Greeks from 447 to 438 BC, is regarded by many people that posses the Golden Ratio in design. The Figure 10 below shows a Golden Rectangle with a Golden Spiral overlaying to the entire face of the Parthenon. This illustrates that the height and width of the Parthenon conform closely to Golden Ratio proportions with an assumption that the bottom of the golden rectangle should align with the bottom of the second step into the structure and that the top should align with a peak of the roof that is projected by the remaining sections. Given that assumption, the top of the columns and base of the roof line are in a close golden ratio proportion to the height of the Parthenon.

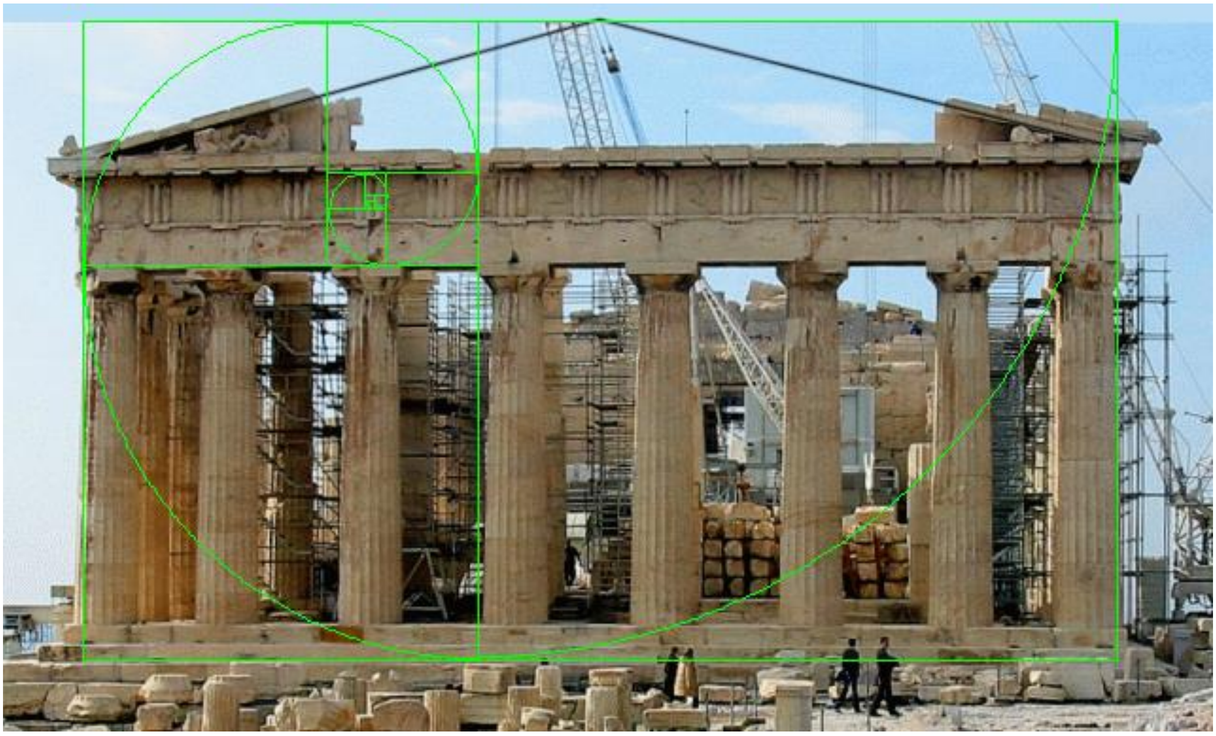


Figure 10 : Parthenon with Golden Rectangle and Spiral.

In the Figure 10, golden ratio is observed on the grid lines to elements of the Parthenon that remaining standing. The grid lines appear to illustrate golden ratio proportions in these design elements in Figure 11:

- **Height of the columns** – The structural beam on top of the columns is in a golden ratio proportion to the height of the columns. Note that each of the grid lines is a golden ratio proportion of the one below it, so the third golden ratio grid line from the bottom to the top at the base of the support beam represents a length that is phi cubed, 0.236, from the top of the beam to the base of the column.
- **Dividing line of the root support beam -** The structural beam on top of the columns has a horizontal dividing line that is in golden ratio proportion to the height of the support beam.
- **Width of the columns** – The width of the columns is in a golden ratio proportion formed by the distance from the center line of the columns to the outside of the columns.

Figure 11 : Grid elements of Parthenon with Golden Ratio.

The Figure 12 below shows the golden ratio proportions that appear in the height of the roof support beam and in the decorative rectangular sections that run horizontally across it. The gold colored grids below are golden rectangles, with a width to height ratio of exactly 1.618 to 1.

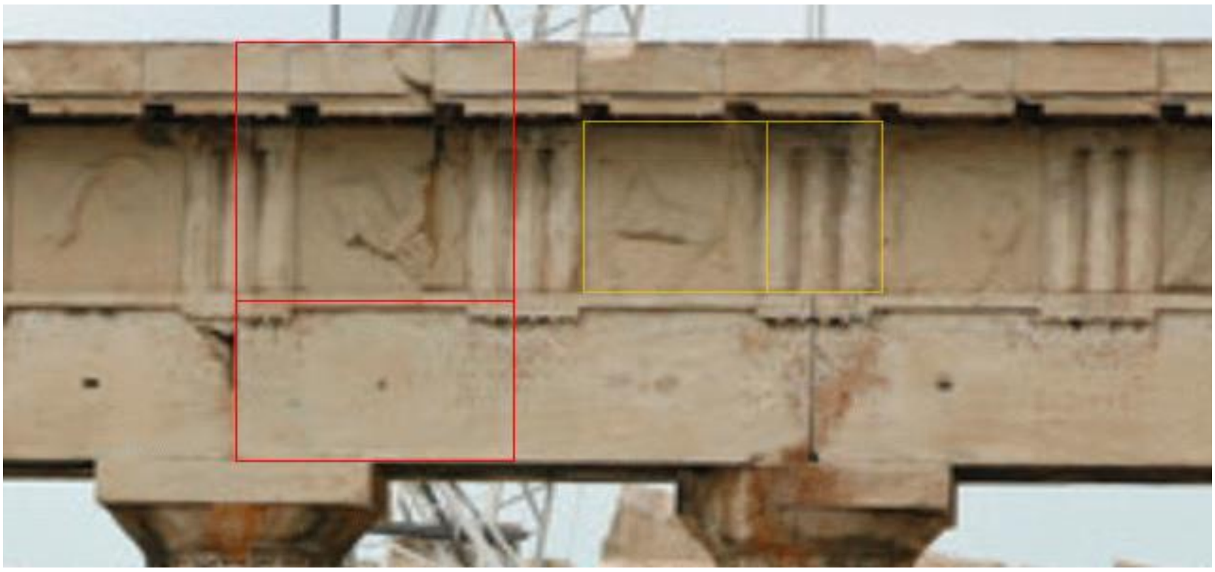


Figure 12 : Golden Ratio at the roof of Parthenon.

3.3. Modern Buildings

Not only ancient ones, but also modern buildings still use Golden Ratio in their construction.

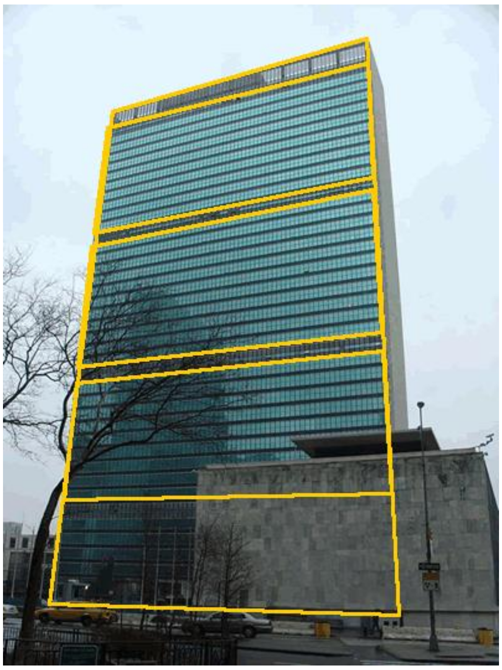


Figure 13 : UN Building.

Figure 14 : CN Tower.

If you take a look to the United Nations Building in New York, you will observe two golden rectangles, one from base to the upper intermediate level, and the second rectangle from the top to the lower intermediate level, as drawn by yellow lines on the Figure 13.

Another example is the world tallest tower, CN Tower shown on Figure 14. CN Tower, located in Toronto, has a height of 553.33 meters. The ratio of observation deck which is at 342th meters to the total height of 553.33 is 0.618 or phi that is the reciprocal of Golden Ratio.

3.4. Islamic World

Figure 15 : Selimiye Mosque.

The great architect Sinan used Golden Ratio in Selimiye Mosque (Figure 15) and Suleymaniye Mosque. The ratio of lighted balconies of the minarets gives Golden Ratio.

Many other examples on architectural constructions of the modern world can be added.

4. FIBONACCI SEQUENCE

The sequence of numbers $\{F_n\}^{\infty}_{n=1}$ defined as The Fibonacci numbers by the linear recurrence equation

$$F_n = F_{n-1} + F_{n-2} \tag{13}$$

With $F_1 = F_2 = 1$ and as a result of the definition, it is conventional to define $F_0 = 0$. Using the equation (13) the Fibonacci numbers for n = 1, 2, ... are calculated as 1, 1, 2, 3, 5, 8, 13, 21, ...

The sum of the squares of any series of Fibonacci numbers is equal the last Fibonacci number which is used in the series times the next Fibonacci number. Thus, based on the progression below and properties of the Fibonacci series, the result is Fibonacci spiral:

$$1^2 + 1^2 + 2^2 + 3^2 + 5^2 = 5 \text{ x } 8 \tag{14}$$

Giving the general formula

$$1^2 + 1^2 + . . . + F_n^2 = F_n \text{ x } F_{(n+1)} \tag{15}$$

Although Golden Spiral is based on a series of identically proportioned golden rectangles, each having a golden ratio of 1.618 of the length of the long side to that of the short side of the rectangle, Fibonacci spiral is very similar to Golden Spiral. The Fibonacci spiral is slightly different than the Golden Spiral for the smaller values of n. As n increases, Fibonacci Spiral gets closer and closer to a Golden Spiral because of the ratio of each number in the Fibonacci series to the one before it converges on Phi, 1.618, as the series progresses (e.g., 1, 1, 2, 3, 5, 8 and 13 produce ratios of 1, 2, 1.5, 1.67, 1.6 and 1.625, respectively).

Fibonacci spirals and Golden spirals appear in nature on many creatures. But not every spiral in nature is related to Fibonacci numbers or Golden Number. Most spirals found in nature are equiangular spirals, whereas Fibonacci and Golden spirals are special cases of the broader class of Equiangular spirals. An Equiangular spiral itself is a special type of spiral with unique mathematical properties in which the size of the spiral increases but its shape remains the same with each successive rotation of its curve. In nature, equiangular spirals occur simply because they result in the forces that create the spiral are in equilibrium, and are often seen in non-living examples such as spiral arms of galaxies and the spirals of hurricanes.

5. MATHEMATICS IN NATURE

Fibonacci spirals, Golden spirals and golden ratio-based spirals generally appear in living organisms, as illustrated below:

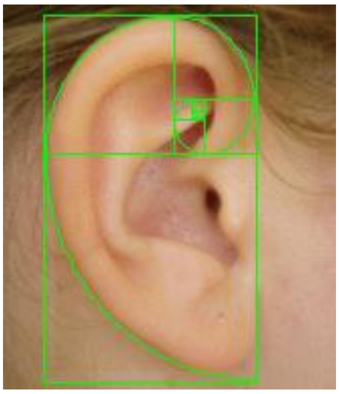


Figure 16 : Human Ear having Golden Spiral

Human ear has a shape of Golden Spiral and Golden Rectangle can easily be drawn on it.

On the other hand, the Nautilus shell which can be found on many parts of the world is often shown as an illustration of the golden ratio in nature, but the spiral of a nautilus shell is not a golden spiral, as the Figure 16 shows.

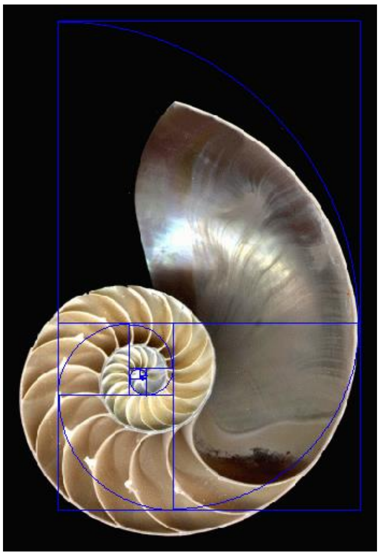


Figure 17 : Nautilus Shell and its spiral.

As seen on the Figure 17 above, Nautilus Shell comprises a Golden Rectangle, however, its spiral is not a Golden Spiral, but equiangular spiral is.

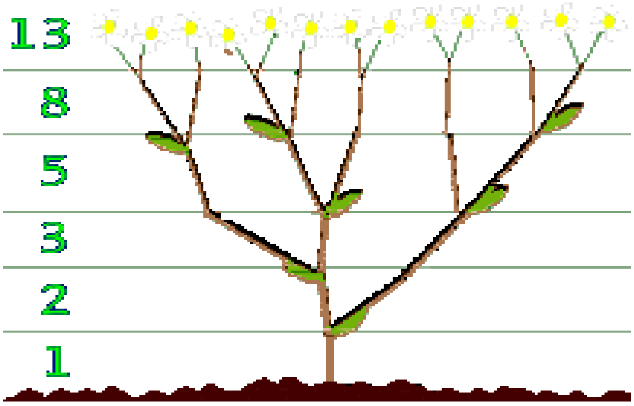


Figure 18 : Fibonacci Numbers on Daisy Flower.

Figure 18 shows that the growth of Daisy Flower obeys the Fibonacci Numbers.

There are many examples of Fibonacci Spirals in nature. Sunflowers locate their seeds in Fibonacci Spirals and reveals Fibonacci Numbers.

Figure 19 : Sunflower and Fibonacci Numbers.

On the other hand, as seen on Figure 19, the number of seeds of the sunflower on clockwise is 55, while it is 89 on counterclockwise. The ratio of these numbers is 89/55 = 1.61818 which is Golden Number.

Figure 20 : Fibonacci Numbers on Pinecone.

Pinecones and pineapples show the same construction as sunflower. The numbers of seeds are arranged in Fibonacci numbers. If you count the seeds of pinecone shown on Figure 20, you will find that the numbers of seeds are 13 on clockwise and 8 in counterclockwise, which are Fibonacci Numbers.

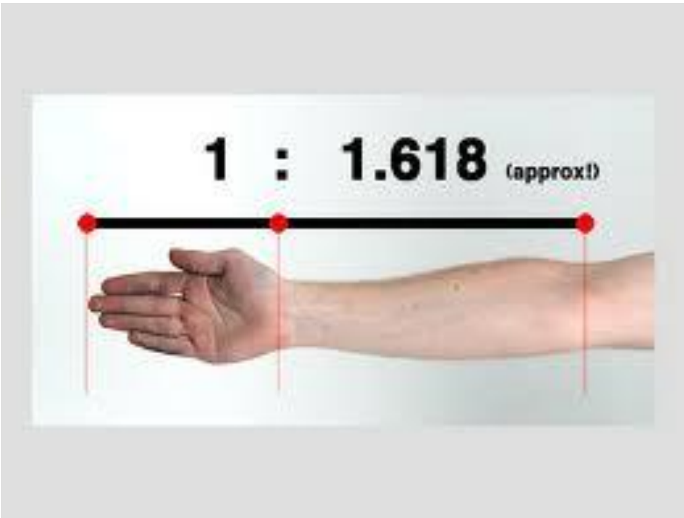


Figure 21 : Golden Ratio in human arm.

The elbow divides the arm into two parts. The ratio of upper part to the lower gives Golden Ratio. On the other hand, the ratio of the lower part to the hand is also a Golden Ratio (Figure 21).

6. GOLDEN RATIO IN PAINTING

As the Golden Ratio is found in the design and beauty of nature, we can say that it can also be used for beauty and balance in the design of art. The use of Golden Ratio in art is not a rule, but it is a tool resulting more aesthetic work. The Golden Rectangle is proposed to be the most aesthetically pleasing of all possible rectangles. For this reason, it and the Golden Ratio have been used extensively in art and architecture for many years. The most prominent and well known uses of the Golden Rectangle in art were created by the great Italian artist, inventor, and mathematician, Leonardo da Vinci.

The Mona Lisa, undisputably Leonardo's most famous painting, is full of Golden Rectangles. If you draw a rectangle whose base extends from the woman's right wrist to her left elbow and extend the rectangle vertically until it reaches the very top of her head, you will have a Golden Rectangle (Figure 22).

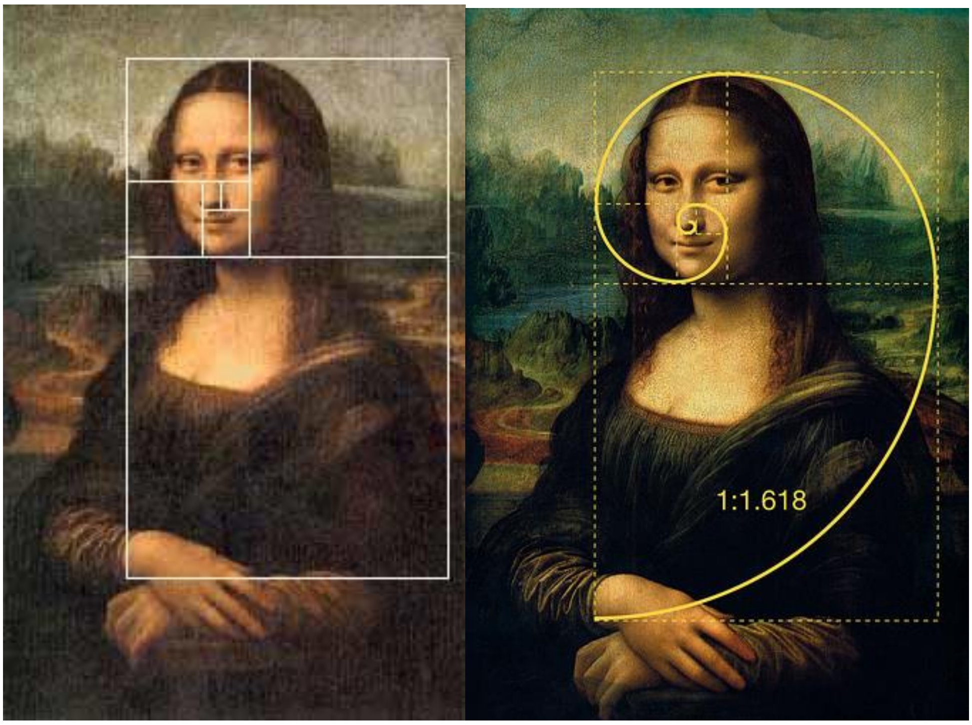


Figure 22 : Mona Lisa, Golden Rectangle and Spiral.

You will discover that the edges of these new squares come to all the important focal points of the woman: her chin, her eye, her nose, and the upturned corner of her mysterious mouth, if you draw squares inside this Golden Rectangle. By drawing a rectangle around her face, we can see that it is indeed golden rectangle. If we divide that rectangle with a line drawn across her eyes, we get another golden rectangle. It is considered that Leonardo, as a mathematician, made this painting line up with Golden Rectangles in this fashion in order to further the incorporation of mathematics into art on purpose. In addition to that, the overall shape of the woman is a triangle with her arms as the base and her head as the tip. This is meant to draw attention to the face of the woman in the portrait.

Leonardo's another famous study of the proportions of man, "The Vetruvian Man" (The Man in Action), is also full of Golden Rectangles. In case of the Mona Lisa, all the lines of the Golden Rectangle are assumed by the mathematicians. But in "The Vetruvian Man", many of the lines of the rectangles are actually drawn into the image. There are three distinct sets of Golden Rectangles in this painting: one set for the head area, one for the torso, and one for the legs. Figure 23 shows these details.

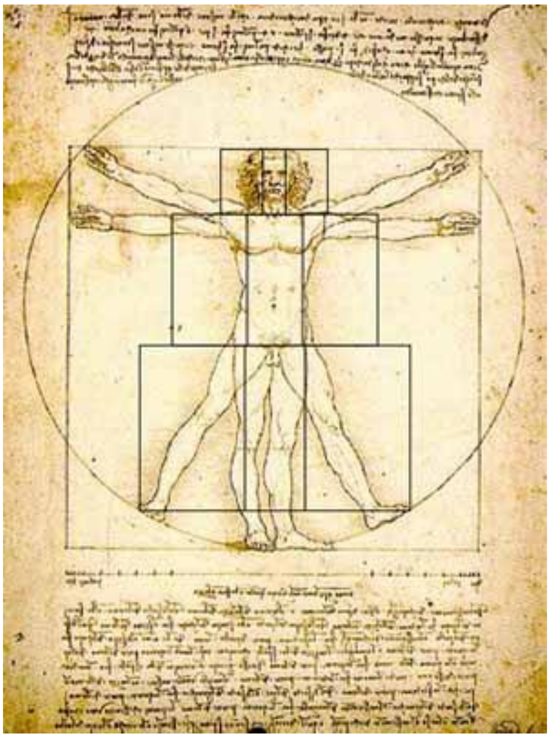


Figure 23 : The Vetruvian Man of Leonrdo Da Vinci.

The famous painting by Leonardo da Vinci, “The Last Supper”, contains a lot of Golden Rectangles. In this painting, successive divisions of each section by the golden section define the key elements of composition. The table, ceiling, people, windows are full of Golden Rectangles shown at the Figure 24 below, and the painting itself is a perfect sample of Golden Ratio.

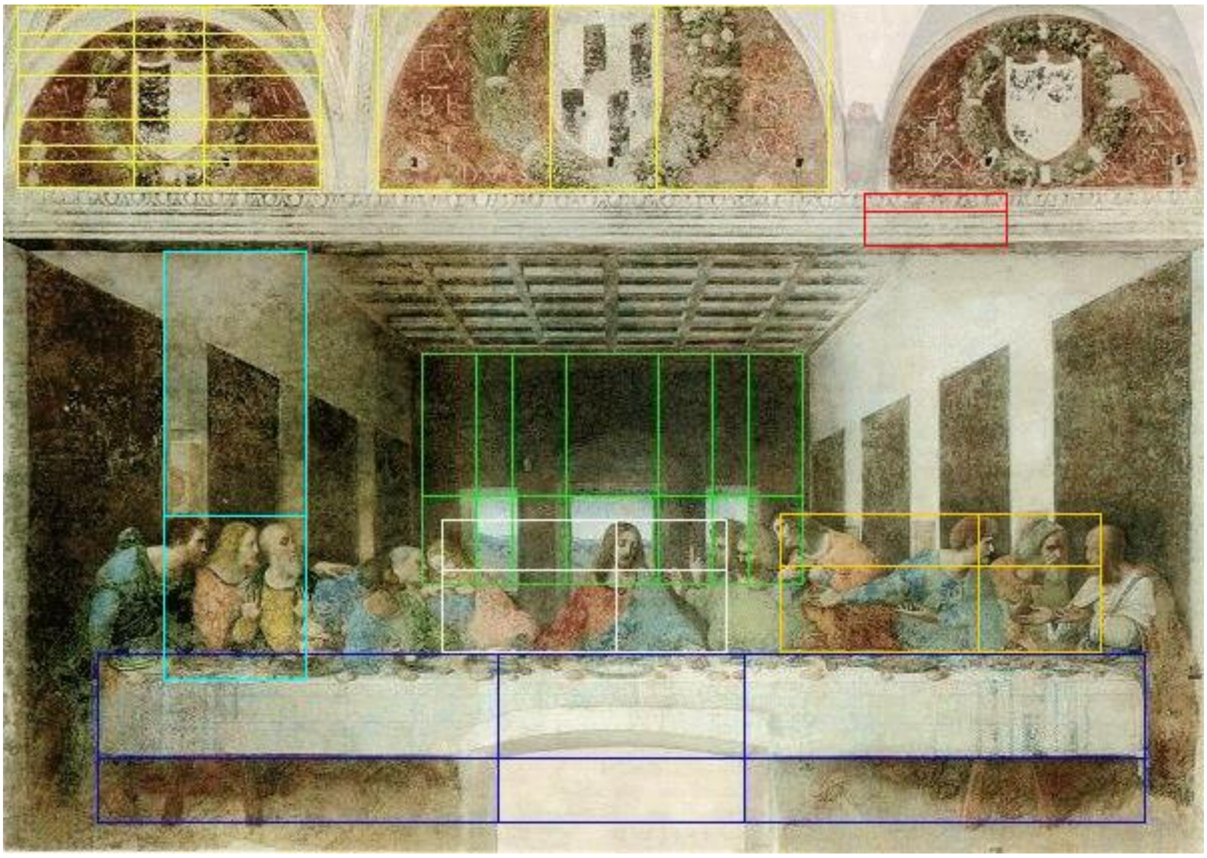


Figure 24 : The Last Supper and Golden Rectangles.

Golden Rectangle was used in their paintings by other famous artists. In “The Sacrament of the Last Supper” (Figure 25), it is seen that Salvador Dali’s painting is framed in a golden rectangle. Dali positioned the table exactly at the golden section of the height of his painting following Da Vinci’s lead. He positioned the two disciples at Christ‘s side at the golden sections of the width of the composition. In addition, the windows in the background are formed by a large dodecahedron which consists of 12 pentagons which has golden ratio relations in their proportion.

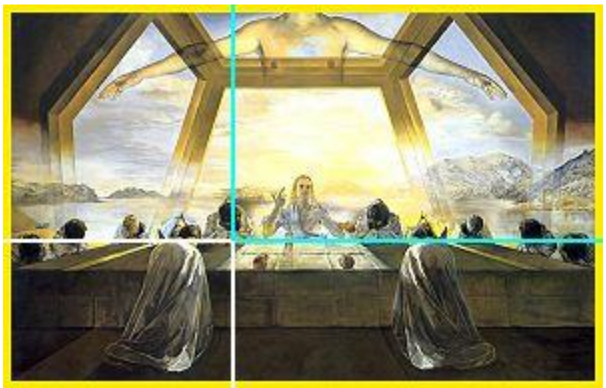


Figure 25 : Salvador Dali's The Sacrament of the Last Supper.

7. MATHEMATICS IN MUSIC

In ancient world, mathematicians interested arts and music also. Pythagoras, famous mathematician, and others studied deeply on music by using mathematics. It is interesting that the other way around is also true. Some musicians are interested in mathematics in their art studies. Within those ancient scientists, Pythagoras stated basic rules of noting and octave.

According to a myth, Pythagoras noticed that the sound rhythm of the hammers of workers had a ratio of 12:9:8:6. Then Pythagoras in his study on music divided a 12 units wire into two parts and obtained octave. With the 6 units wire (half of the whole) is high-pitched of 12 units octave. Pythagoras discovered 5 interval system with 8units of a wire which is the 2/3 of the original wire, and 4 interval system with 9 units of a wire which is ¾ of original wire. In ancient world, hearing four voices at a time was called "tetra cord" which assumed the base of music theory. Thus by using 6, 8, 9 and 12 units wires tetra cord was obtained. These numbers have close relations with the golden ratio. In accordance with Pythagorian ratios, the difference between 5 interval system and 4 interval system is a full tone. (3/2 :4/3 = 9/8 ; 5T – 4T = 2M). That is, multiplying a full sound with 9/8 gives a high-pitched of that sound. If we continue, 9/8 x 9/8 = 81/64 (2M + 2M = 3M). Suppose that we start with the note Do. ½ of Do gives one octave pitched of Do. 3/2 of Do gives Sol, 4/3 gives Fa, 9/8 gives Re, 81/64 gives Mi notes. The other intervals can be written as

4/3 : 9/8 = 32/27	4T – 2T = 3m
2/1 : 32/27 = 27/16	6M
2/1 : 81/64 = 81/32	6m
2/1 : 7/8 = 16/9	7m

In this way, we obtain 1/1, 9/8, 81/64, 4/3, 3/2, 27/32, 243/128 and 2/1 for the notes Do, Re, Mi, Fa, Sol, La, Si and Do respectively.

Pythagoras obtained a full tone with 8/9 th of a wire. Starting with the basic ratio of 3/2 (full 5), it was discovered that if this full tone added 12 times continuously to a note, then 7 octave higher of that note is obtained. This is called Pythagorian Comma. In time, Pythagorian System changed so that 12 equal half-toned is achieved. In this system, a full tone is represented with to half tones instead of 3/2.

$$(3/2)^{12}/2^{7}= 1.0136432 = (9/8)^{6}/2 = \text{Pythagorian Comma.} \qquad (16)$$

Tempered 5 is designated by 7 half-tone which is narrower than Pythagorian 5 interval. Tempered 4 is designated by 5 half-tone which is greater than Pythagorian 4 interval. Studies showed that human ear is more sensitive to Pythagorian intervals, but it is impossible to quit tempered intervals anymore.

Composers used Golden Ratio and Fibonacci Numbers in their work of art. Bèla Bartòk, a famous composer of 20th century, is a leader artist in this respect.

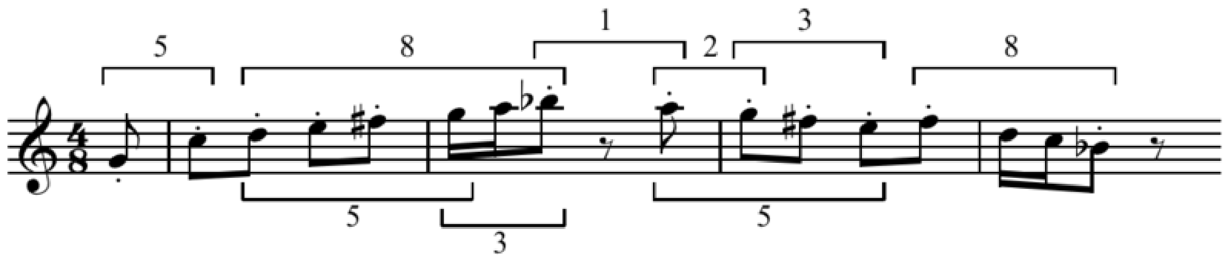


Figure 26 : Fibonacci Numbers in Bartòk's Sonata.

Bartòk's Sonata for Two Pianos and Percussion, as shown in Picture 26 above, contains Fibonacci Numbers in intervals. Composer's other art works also contains Fibonacci Numbers and Golden Ratio.

Figure 27 : Bartòk's Music for Strings, Percussions and Celesta.

Bartòk's Music for Strings, Percussions and Celesta third movement is based on the Fibonacci sequence as this "written-out accelerando/ritardando" uses the rhythm 1:2:3:5:8:5:3:2:1 (Figure 27).

Another very well known example is George Frideric Handel's Hallelujah (or Messiah) which has 94 measures in total. One of the most important parts is "Kings of Kings", the entry of solo trumpets that starts at 57 th and 58 th measures, i.e. 8/13 th of 94 measures (94 x 8/13 ≈ 58). "The Kingdom of Glory" theme starts at 34 th measure (remember that 34 is a Fibonacci Number) which is 8/13 th of the first 57 measures. At the 8/13 th measure of the second 37 measure (=22), that counts up tp 79 th measure, "And he shall ..." theme starts. These samples reveal that Golden Ratio and Fibonacci Numbers can be used in musical art.

8. CONCLUSION

The above explanations, examples and proofs show that mathematics and art are inevitably interrelated. Not only art, but nature and universe reveal mathematical forms very clearly. Mathematics is not an abstract science, but its effects can be seen everywhere and in every living organism. Mathematics is based on truth and proof, whereas art is based on thoughts and imagination. But a wide imagination requires a wide angle of sight, which can be gained by mathematics. All explanations given above study clearly indicate that art and mathematics are very closely related to each other and art without mathematics cannot be considered alone. On the other hand, mathematics itself is art with its magnificent applications. Thus, we come to the conclusion that mathematics and art constitute an inseparable composition. Understanding and enjoying the world we live in, we have to understand mathematics and art, and their undeniable relation and cooperation.

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