TED ANKARA COLLEGE FOUNDATION PRIVATE HIGH SCHOOL

Determining a Correlation Between Financial Risk and Expected Return in Economics by Using Portfolio Optimization

Mathematics Extended Essay

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Session: May 2012 Supervisor: Mrs. Beril Ayden Word Count:3143

Abstract

Investing at the stock market is often considered as a way of gambling. That is because most people can't manage the see the mathematics behind it. Someone with enough mathematical knowledge can see the algorithms in movements of stocks and find the correct function for it, basically optimize their portfolio. With the correct function one can foresee fate of their investments or when to invest on what. The aim of portfolio optimization is to find the set of efficient feasible portfolios. A portfolio is feasible if it satisfies a relevant set of relevant linear constraints; it is efficient if it provides less risk than any other feasible portfolio with the same expected return and more expected return than any other feasible portfolio with same risk. Thus, the research question of this Mathematics Extended Essay, "Is it possible to determine a correlation between financial risk and expected return in economics by using portfolio optimization?"arises.

In order to answer the question first a scenario had to be selected. The scenario chosen for this extended essay was the Turkish Stock Market (more commonly known as its Turkish abbreviation IMKB) from January 2009 to September 2011. After that the statistical data needed for domains of functions is gathered from the Central Bank database. Then the objective function is created. To create the objective function, first the risk function had to be created. Portfolio risk is stated in terms of absolute deviation of rate of return. Risk function is a linear combination of the two semi-absolute deviations of return from the mean. (Spenza, 1993) The objective function is the function that minimizes the risk function depending on the nature of the investor, i.e. for a risk seeking investor higher interest rate investments would be chosen. To select the optimal portfolio, instead of Markowitz Model, a linear programming model (a model where functions are created as linear equations) is used because of computational difficulties caused by quadratic nature of the Markowitz Model. To solve the model LINDO optimization modeling software is used since the model contained 32 linear equations with 5 variables. At the end 12 different portfolios with different interest rates are produced for different types of investors.

Word Count: 363

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The business schools reward difficult complex behaviour more than simple behaviour, but simple behaviour is more effective. – Buffet, Warren

1. Introduction

1.1 Why did I choose a stock market to model upon?

Stocks' movement at the market is often considered confusing by the people. This public misconception usually held the small investor back by convincing them that they need to take risks they can't overcome, in order to make profit. What the small investor doesn't know is that every stock's value can be expressed as a compound function, a combination of other functions and the daily changes of a stock value can be expressed mathematically. Thus, an investor's portfolio becomes only a mathematical function where the aim is minimizing risk, making it a perfect candidate for modeling upon. This selection of assets to satisfy certain criteria, such as calculating expected amount of return for a given amount of portfolio risk or minimizing portfolio risk for a given amount of return, is called portfolio optimization.

1.2 What is optimization?

Optimization is selecting the best available output from the solution set of given function f(x) by either maximizing or minimizing it. Here is an example of optimization model, given by George B. Dantzig, considered as the father of mathematical programming.[1]

- f(x) is the objective function
- X is the domain of function f(x) where x X
- Functions g(x) and h(x) are called constraints

The maximization of f(x) is achieved by minimization of constraints. Therefore the following equation is reached:

$$\text{Maximize } f(x): x \text{ in } X,\ g(x) <= 0,\ h(x) = 0$$

A point x is feasible if it is in domain X and satisfies the constraints: g(x) <= 0 and h(x) = 0. A point x' is optimal if it is feasible and if the value of the objective function is not less than that of any other feasible solution, making the optimal solution a subset of all feasible solutions.

1.3 What is portfolio optimization?

Portfolio optimization is determining which assets to be selected from a set of other financial instruments to form the investor's portfolio by meeting certain criteria, defined as an efficient

[1] Mathematical Programming Glossary. (10.12.2011)
http://glossary.computing.society.informs.org/index.php?page=nature.html

portfolio by Harry Markowitz.[2] "Modern Portfolio Theory (MPT) is based on a simple assumption that risk is defined by volatility. According to the theory, investors are risk adverse: they are willing to accept more risk (volatility) for higher payoffs and will accept lower returns for a less volatile investment."[3] With the correct equation any investor can select the optimal portfolio depending on their personality, allowing a low risk portfolio for a more laid back investor or a more profiting one for a more ambitious investor. Since every portfolio is personal every optimization equation has to be different than the other one. Nevertheless, by using the same model and a wider domain a pattern can be found between two different variables. Hence, the research question of this essay;

"Is it possible to determine a correlation between financial risk and expected return in economics by using portfolio optimization?"

can be answered forming the hypothesis;

"It is possible to determine a correlation between financial risk and expected return in economics by using portfolio optimization to model on a set of given assets as domain."

2. Data Collection

Domain or data sets for this essay will be obtained from government offices Capital Markets Board of Turkey (more commonly known as its Turkish abbreviation SPK) and Central Bank.

The data sets, which will be referred as probability variables, used are:

- X_1: Gold exchange rate
- X_2: US Dollar exchange rate
- X_3: Euro exchange rate
- X_4: Istanbul Stock Exchange Stock Index, for time interval January 2009 – September 2011

Date	Gold	US Dollar	Euro	Istanbul Market Stock Exchange Index
January 2009	44,0	1,60	2,125	24.963
February 2009	50,2	1,66	2,126	24.114

[2] Markowitz, H.M. (March 1952). "Portfolio Selection". *The Journal of Finance* **7** (1): 77–91

[3] Modern Portfolio Theory Criticisms (n.d.) in Travis Morien's investment FAQ. Retrieved from http://www.travismorien.com/FAQ/portfolios/mptcriticism.htm

March 2009	51,5	1,71	2,229	28.491
April 2009	46,1	1,61	2,127	33.853
May 2009	46,5	1,56	2,122	35.244
June 2009	47,4	1,55	2,169	38.316
July 2009	45,8	1,52	2,142	45.479
August 2009	45,8	1,49	2,119	46.436
September 2009	48,1	1,49	2,168	49.851
October 2009	49,5	1,47	2,177	47.015
November 2009	53,2	1,49	2,216	50.348
December 2009	54,9	1,51	2,205	54.247
January 2010	52,8	1,47	2,107	54.467
February 2010	53,6	1,51	2,074	51.934
March 2010	55,2	1,54	2,086	53.718
April 2010	55,2	1,50	2,009	58.648
May 2010	60,5	1,54	1,949	55.592
June 2010	62,4	1,58	1,927	55.585
July 2010	59,5	1,54	1,965	58.156
August 2010	59,5	1,51	1,951	59.218
September 2010	61,6	1,496	1,86	63.180
October 2010	61,6	1,425	1,984	68.787
November 2010	63,9	1,437	1,972	68.599
December 2010	68,0	1,52	2,018	66.037
January 2011	68,3	1,561	2,082	66.735
February 2011	70,5	1,59	2,17	64.354
March 2011	72,6	1,582	2,213	62.940
April 2011	72,5	1,523	2,197	68.123
May 2011	76,5	1,571	2,26	65.645
June 2011	78,9	1,602	2,3	62.591
July 2011	83,7	1,651	2,36	62.435
August 2011	100,1	1,752	2,512	54.597
September 2011	102,7	1,791	2,476	57.292

Table 1: Monthly exchange rates of gold, US Dollar, Euro and Istanbul Stock Market Index against Turkish Lira

3. Assumptions of the Model

In order to do the necessary calculations Modern Portfolio Theory makes assumptions regarding the market that is modeled upon. Assumptions are crucial for the model because every assumption made omits another function or human factor from the model allowing a sample portrait of the market. However, these assumptions are often criticized and caused the applicability of MPT to the real markets for investment and financial gains to be questioned. These assumptions are:

- Investors are only interested in the optimization of the problem given in the model.
- Asset returns are normally distributed random variables
- Assets have a constant correlation between each other for the given time interval meaning they move together
- All investors try to profit as much as possible
- All investors are rational and risk averse. Meaning that all investors will choose the asset with maximum and will go after higher risk for higher return
- All investors have same amount of information at the same time, meaning that no matter how many assets are traded on the market and how dynamic they are investor will always know which asset returns what and invest according to it.
- Investors accurately predict possible returns of assets. In other words expected values expected by the investors is always true allowing investors to invest for exactly what they want
- There aren't any taxes and transactions are free so that investors won't lose money while trading. However, this assumption can be also be true for certain situations which contain investors with same amount of capital who make same amount of transaction each time at the same market. As a result, they will pay the same taxes and same transaction fees which can then become negligible.
- Investors do not affect prices. Market activity or amount of shares an investor buys do not change
- Investors can trade unlimitedly if there is no risk
- Assets can be divided in any proportion while buying and selling

4. Deduction of Objective Function

In order to provide optimum portfolio for the investor, absolute deviation method is used instead of standard deviation to measure risk value of net profit. The investors' risk is estimated. By doing so, instead of creating a quadratic probability or a solution set of risk estimations, a diagonal equation set is used. As a result, the purpose function will minimize the risk function.

The risk function is;

$$\frac{\sum_{t=1}^{T} \left| \sum_{j=1}^{n} (r_{jt} - r_j)\, X_j \right|}{T}$$

Where;

j = Set of stocks

t = time interval

T = number of time intervals examined

r_j = average profit ratio of stock j

X_j = Share of investments belonging to stock j

r_{jt} = is the average profit ratio of stock j observed during one t time interval

Risk function is assumed as objective function in this model. Objective function is minimized under following constraints:

The ρ represents the expected rate of returns of portfolio. The C represents the total investment funds including various assets.

The share of each asset in portfolio must be equal to the total investment fund.

We may assign upper limit (U) of any asset. For example if the investor wants to exceed the share of bond 10 percent. She/he should incorporate the constraints as follows;

The portfolio optimization that used in the "Modern Portfolio Theory" by Harry Markowitz[4] was a quadratic equation. It was derived from standard deviation.

Where;

E {X} = the average value of probability variable X

R (x) = the probability variable, which represents the profit of a portfolio calculated by using x variables

$$R(X) = \sum_{j}^{n} R_j X_j \qquad j=1,\ldots,n$$

μ (x) = the arithmetic mean of probability variable R (x)

$$\mu(x) = E\left(\sum_{j}^{n} R_j X_j \right)$$

In the Markowitz model the objective function to be minimized is:

$$\sigma(x) = \sqrt{\sum_{i} \sum_{j} \sigma_{ij} X_i X_j} \qquad \begin{matrix} i = 1, \ldots, n \\ j = 1, \ldots, n \end{matrix}$$

Where;

n = total number of assets

[4] Markowitz, H.M. (March 1952). "Portfolio Selection". *The Journal of Finance* **7** (1): 77–91

This function will be named as L_1.

Over the time as processing capacities of computers and academic resources on the subject increased, alternative methods to select the optimal portfolio are created. Since the objective function of the Markowitz model is a quadratic equation, the portfolio optimization problem is handled as a quadratic programming problem. Since it is easier to solve a linear problem rather than a quadratic one, a different model named after Hiroshi Konno and Hiroaki Yamazaki.[5] The objective function of Konno & Yamazaki Model which uses absolute deviation is called L_2.

$$w(x) = E\left[\,\left|\sum_{j=1}^{n} R_j X_j - E\left(\sum_{j=1}^{n} R_j X_j\right)\right|\,\right]$$

Multivariate normal distribution of return rates ($R_1, \ldots, R_n$) which are probability variables is[6]:

$$w(x) = \sqrt{2/\Pi}\ \sigma(x)$$

Where, by minimizing L_2, absolute deviation, risk function (w(x)), L_1, standard deviation, risk function (σ(x)) is minimized.

Notation used in this model in which portfolio risk is minimized is as following:

j = Set of assets

C = Amount of total investment

p = Ratio of expected return

U_j = Top limit of investment for asset j

R_j = Return rate of asset j and probability variable of the model

T = Time intervals being inspected

r_j = Average return rate of asset

[5] Konno, H., Yamaazaki H. “Mean Absolute Deviation Portfolio Optimization Model and Its Applications to Tokyo Stock Market”. Management Science, Vol. 37, No. 5 (May, 1991), 519-531

[6] R ao, C.R. (1965). Linear Statistical Inference and Applications. New York: John Wiley & Sons Inc.

X_j = Portion of the investment belonging to asset j

Portfolio is selected by means of decision variables, X_j

R(x), which is the probability variable, represents return of portfolio calculated by using variable x.

$$R(X) = \sum_{j}^{n} R_j X_j \qquad j = 1,\ldots,n$$

Average value of probability variable R(x) is notated as μ(x) and shown as following

)

By using the information given above Konno and Yamazaki Model can be obtained from the Markowitz Model.

$$M(x) = E\{[R(x) - \mu(x)]^2\}$$

E { x } represents average value of probability variable X. At Markowitz Model, function L_1 is the objective function which needs to be minimized. Since the objective function is a quadratic equation at the Markowitz Model, portfolio optimization problem is handled as a quadratic programming problem. Objective function for n number of assets is:

$$\min \sum_{i}^{n} \sum_{j}^{n} \sigma_{ij} X_i X_j$$

Constraints of the objective function are composed of linear equations.

$$\sum_{j}^{n} r_j X_j \geq \rho C$$

$$\sum_{j}^{n} X_j = C$$

$$0 \leq X_j \leq U_j \qquad j \in J$$

Since absolute deviation is used instead of standard deviation in the Konno and Yamazaki Model, function K(x) can be written instead of function M(x) of Markowitz Model.

$$K(x) = E\{ | R(x) - \mu(x) | \}$$

Average value of r_{jt} return observed for all t time intervals of data set $\{r_{jt} \text{ I } t = 1, \dots, T\}$ is r_j.

$$r_j = \frac{\sum_{t=1}^{T} r_{jt}}{T}$$

When this equation is used in the one above the following equation is obtained.

$$\frac{\sum_{t=1}^{T} \left| \sum_{j=1}^{n} (r_{jt} - r_j) X_j \right|}{T}$$

In order to express the above function simply function y_t can be defined as following:

$$y_t = \left| \sum_{j=1}^{n} (r_{jt} - r_j) X_j \right| \qquad t = 1 \ldots T$$

By using the above equation the linear programming problem of portfolio optimization for Konno and Yamazaki Model can be given as following:

$$\min \frac{\sum_{t=1}^{T} y_t}{T}$$

$$y_t + \sum_{j=1}^{n} (r_{jt} - r_j)\, X_j \geq 0 \qquad t = 1, \ldots, T$$

$$y_t - \sum_{j=1}^{n} (r_{jt} - r_j)\, X_j \geq 0 \qquad t = 1, \ldots, T$$

$$\sum_{j=1}^{n} r_j X_j \geq \rho C$$

$$\sum_{j=1}^{n} X_j = C$$

$$j \in J$$

In order to determine every point of the activity limit of Konno & Yamazaki Model linear programming problem containing constraints consisting of decision variables and 2T + 2 needs to be solved.

5. Equation

The objective function in the model is minimization of sum of function y_t calculated for each time interval, t. The function y_t is the absolute value of the result where the values obtained by subtracting the return rates of stocks from average return in time interval t are used as coefficients. Monthly return rate is calculated by subtracting given assets' monthly exchange value from previous month's value. Since there isn't any previous data monthly return for time interval t=1 (January 2009) is 0.

Date	Gold	US Dollar	Euro	Istanbul Market Stock Exchange Index
January 2009				
February 2009	14,05	3,98	0,03	-3,40
March 2009	2,61	3,16	4,87	18,15
April 2009	-10,55	-5,99	-4,59	18,82
May 2009	1,04	-3,16	-0,25	4,11
June 2009	1,83	-0,77	2,22	8,72
July 2009	-3,44	-1,70	-1,24	18,69
August 2009	0,09	-2,28	-1,09	2,11
September 2009	4,98	0,41	2,33	7,35
October 2009	2,85	-1,56	0,42	-5,69
November 2009	7,64	1,23	1,79	7,09
December 2009	3,21	1,32	-0,51	7,75
January 2010	-3,93	-2,21	-4,42	0,41
February 2010	1,46	2,68	-1,59	-4,65
March 2010	3,08	1,53	0,56	3,44
April 2010	0,04	-2,67	-3,67	9,18
May 2010	9,56	3,15	-2,99	-5,21
June 2010	3,19	2,31	-1,11	-0,01
July 2010	-4,63	-2,16	1,96	4,63
August 2010	-0,07	-2,26	-0,71	1,83
September 2010	3,58	-0,85	-4,67	6,69
October 2010	-0,11	-4,75	6,67	8,87
November 2010	3,75	0,84	-0,60	-0,27
December 2010	6,53	5,78	2,33	-3,73
January 2011	0,43	2,70	3,17	1,06
February 2011	3,19	1,86	4,23	-3,57
March 2011	2,91	-0,50	1,98	-2,20
April 2011	-0,08	-3,73	-0,72	8,23
May 2011	5,54	3,15	2,87	-3,64
June 2011	3,07	1,97	1,77	-4,65

July 2011	6,12	3,06	2,61	-0,25
August 2011	19,59	6,12	6,44	-12,55
September 2011	2,60	2,23	-1,43	4,94

Table 2: Monthly return rates of gold, US Dollar, Euro and Istanbul Stock Market Index against Turkish Lira

By dividing the sum of monthly return rates with number of time intervals average return is found.

Asset	Gold	US Dollar	Euro	Istanbul Market Stock Exchange Index
Average Return	2,8	0,4	0,5	2,9

Table 3: Average return of gold, US Dollar, Euro and Istanbul Market Stock Exchange Index against Turkish Lira calculated for time interval January 2009 – September 2011 based on monthly return rates

By subtracting the return rates of stocks from average return coefficients are obtained.

X1	X2	X3	X4
11,23	3,58	-0,49	-6,28
-0,21	2,76	4,35	15,27
-13,36	-6,39	-5,11	15,94
-1,77	-3,57	-0,77	1,23
-0,99	-1,17	1,70	5,83
-6,26	-2,10	-1,76	15,81
-2,73	-2,68	-1,61	-0,78
2,16	0,00	1,81	4,47
0,03	-1,96	-0,10	-8,57
4,83	0,82	1,27	4,21
0,40	0,91	-1,03	4,86
-6,75	-2,62	-4,94	-2,48
-1,36	2,27	-2,11	-7,53
0,26	1,13	0,04	0,55
-2,78	-3,07	-4,19	6,30
6,75	2,75	-3,51	-8,09

0,37	1,91	-1,63	-2,90
-7,45	-2,57	1,44	1,75
-2,88	-2,66	-1,23	-1,06
0,76	-1,26	-5,20	3,81
-2,93	-5,15	6,15	5,99
0,94	0,44	-1,13	-3,16
3,71	5,37	1,81	-6,62
-2,39	2,29	2,65	-1,82
0,37	1,46	3,71	-6,45
0,09	-0,91	1,46	-5,08
-2,90	-4,13	-1,24	5,35
2,73	2,75	2,35	-6,52
0,25	1,57	1,25	-7,53
3,31	2,66	2,09	-3,13
16,78	5,72	5,92	-15,44
-0,22	1,82	-1,95	2,05

Table 4: Coefficients obtained by subtracting monthly return rates from average return for all data sets.

5.1 The Objective Function

The objective function will be[7]:

MIN Y1+ Y2+ Y3+ Y4 + Y5 + Y6 + Y7 + Y8 + Y9 + Y10 + Y11 + Y12 + Y13+ Y14+Y15 + Y16 + Y17 +Y18 + Y19 + Y20 + Y21 + Y22+ Y23+ Y24 + Y25 + Y26 + Y27 + Y28 + Y29 + Y30 + Y31 + Y32

When coefficients are placed the objective function will be as following for portfolio return of 1 percent;

Y1+11.2288577563287X1+3.5810880731241X2-0.486882327677074X3-6.28359902089073X4>=0

Y2-0.205994955726115X1 +2.75525569566125X2+4.35257527191012X3+15.2701840671622X4>=0

[7] In order for the optimization software to understand the function instead of common mathematical notation a slightly different notation is used. (i.e. instead of ≤, =< sign is used)

Y3-13.3623338254436X1-6.3909461485024X2-5.10616998354501X3+15.9379155238882X4>=0

Y4-1.77447781387621X1-3.56619771931769X2-0.77498514243326X3+1.22818824316281X4>=0

Y5-0.990210884527672X1-1.17426749013186X2+1.69586852472928X3+5.83351609653583X4>=0

Y6-6.25614099676462X1-2.09755748409823X2-1.75583784675805X3+15.8118690344945X4>=0

Y7-2.72918420171329X1-2.67984992691808X2-1.61317666842054X3-0.776296732151351X4>=0

Y8+2.16156914973877X1+0.00408149688247461X2+1.81071755841377X3+4.47236111194585X4>=0

Y9+0.0328208481453469X1-1.95826722737302X2-0.0983572268768X3-8.57018628605707X4>=0

Y10+4.82748814515116X1+0.822547169920346X2+1.27028162360409X3+4.20635994216315X4>=0

Y11+0.395877379575975X1+0.913446436492033X2-1.02832999361969X3+4.86362054312995X4>=0

Y12-6.74815849280261X1-2.61540854063552X2-4.94224613432732X3-2.47666867920177X4>=0

Y13-1.35771084748132X1+2.27366460941226X2-2.10994950454017X3-7.53305689648655X4>=0

Y14+0.264635703871334X1+1.12850710278634X2+0.0403268389047247X3+0.554776640585484X4>=0

Y15-2.78036490506791X1-3.06805319827223X2-4.18927150282422X3+6.29562850296207X4>=0

Y16+6.74515619892192X1+2.75211518813204X2-3.51403015696371X3-8.09257747890153X4>=0

Y17+0.373485855502224X1+1.90939488957516X2-1.62710091612526X3-2.89543036027865X4>=0

Y18-7.44578147597515X1-2.5656251799615X2+1.44224312926674X3+1.74512733824753X4>=0

Y19-2.88377851569625X1-2.66033966184097X2-1.22908166964283X3-1.0554596728069X4>=0

Y20+0.763235143647238X1-1.25582159990765X2-5.19505095600899X3+3.80875953616019X4>=0

Y21-2.93017783731669X1-5.1484479477421X2+6.14612773494542X3+5.99253526867577X4>=0

Y22+0.935839858047609X1+0.439646620228623X2-1.12537764139867X3-3.1555000688038X4>=0

Y23+3.7122900121893X1+5.37346341691763X2+1.8121182690901X3-6.61606287508871X4>=0

Y24-2.39037691846156X1+2.29490977812337X2+2.65091795628668X3-1.82427315164872X4>=0

Y25+0.373802742558597X1+1.45532482920399X2+3.70616615953717X3-6.45086278838768X4>=0

Y26+0.0907922337077522X1-0.905603297017315X2+1.46102788855525X3-5.07884947755721X4>=0

Y27-2.89928697655697X1-4.13191502725291X2-1.24353938359653X3+5.35300189095548X4>=0

Y28+2.72823079708101X1+2.74921568405696X2+2.34700772280764X3-6.51864043926107X4>=0

Y29+0.254495735704234X1+1.57080679309876X2+1.24937257270353X3-7.53346804585895X4>=0

Y30+3.30740473236519X1+2.656218011253X2+2.08815672045267X3-3.13222935389209X4>=0

Y31+16.7771905465969X1+5.71504589977213X2+5.92013903438046X3-15.4351978173746X4>=0

Y32-0.219194191723278X1+1.82356875433101X2-1.95365995082953X3+2.05451540457836X4>=0

Y1-11.2288577563287X1-3.5810880731241X2+0.486882327677074X3+6.28359902089073X4>=0

Y2+0.205994955726115X1-2.75525569566125X2-4.35257527191012X3-15.2701840671622X4>=0

Y3+13.3623338254436X1+6.3909461485024X2+5.10616998354501X3-15.9379155238882X4>=0

Y4+1.77447781387621X1+3.56619771931769X2+0.77498514243326X3-1.22818824316281X4>=0

Y5+0.990210884527672X1+1.17426749013186X2-1.69586852472928X3-5.83351609653583X4>=0

Y6+6.25614099676462X1+2.09755748409823X2+1.75583784675805X3-15.8118690344945X4>=0

Y7+2.72918420171329X1+2.67984992691808X2+1.61317666842054X3+0.776296732151351X4>=0

Y8-2.16156914973877X1-0.00408149688247461X2-1.81071755841377X3-4.47236111194585X4>=0

Y9-0.0328208481453469X1+1.95826722737302X2+0.0983572268768X3+8.57018628605707X4>=0

Y10-4.82748814515116X1-0.822547169920346X2-1.27028162360409X3-4.20635994216315X4>=0

Y11-0.395877379575975X1-0.913446436492033X2+1.02832999361969X3-4.86362054312995X4>=0

Y12+6.74815849280261X1+2.61540854063552X2+4.94224613432732X3+2.47666867920177X4>=0

Y13+1.35771084748132X1-2.27366460941226X2+2.10994950454017X3+7.53305689648655X4>=0

Y14-0.264635703871334X1-1.12850710278634X2-0.0403268389047247X3-0.554776640585484X4>=0

Y15+2.78036490506791X1+3.06805319827223X2+4.18927150282422X3-6.29562850296207X4>=0

Y16-6.74515619892192X1-2.75211518813204X2+3.51403015696371X3+8.09257747890153X4>=0

Y17-0.373485855502224X1-1.90939488957516X2+1.62710091612526X3+2.89543036027865X4>=0

Y18+7.44578147597515X1+2.5656251799615X2-1.44224312926674X3-1.74512733824753X4>=0

Y19+2.88377851569625X1+2.66033966184097X2+1.22908166964283X3+1.0554596728069X4>=0

Y20-0.763235143647238X1+1.25582159990765X2+5.19505095600899X3-3.80875953616019X4>=0

Y21+2.93017783731669X1+5.1484479477421X2-6.14612773494542X3-5.99253526867577X4>=0

Y22-0.935839858047609X1-0.439646620228623X2+1.12537764139867X3+3.1555000688038X4>=0

Y23-3.7122900121893X1-5.37346341691763X2-1.8121182690901X3+6.61606287508871X4>=0

Y24+2.39037691846156X1-2.29490977812337X2-2.65091795628668X3+1.82427315164872X4>=0

Y25-0.373802742558597X1-1.45532482920399X2-3.70616615953717X3+6.45086278838768X4>=0

Y26-0.0907922337077522X1+0.905603297017315X2-1.46102788855525X3+5.07884947755721X4>=0

Y27+2.89928697655697X1+4.13191502725291X2+1.24353938359653X3-5.35300189095548X4>=0

Y28-2.72823079708101X1-2.74921568405696X2-2.34700772280764X3+6.51864043926107X4>=0

Y29-0.254495735704234X1-1.57080679309876X2-1.24937257270353X3+7.53346804585895X4>=0

Y30-3.30740473236519X1-2.656218011253X2-2.08815672045267X3+3.13222935389209X4>=0

Y31-16.7771905465969X1-5.71504589977213X2-5.92013903438046X3+15.4351978173746X4>=0

Y32+0.219194191723278X1-1.82356875433101X2+1.95365995082953X3-2.05451540457836X4>=0

2.8 X1 +0.4 X2 + 0.5 X3 + 2.9 X4 >= 1

X1 + X2 + X3 + X4 = 1

5.2 Solving the Equation

Since an equation of this size is impossible for a high school student to solve in relatively quick fashion, mathematical optimization software is needed. For this equation LINDO mathematical optimization software is used. The optimization program gives a solution set for the equation.

VARIABLE	VALUE
Y1	0.134841
Y2	6.459.194
Y3	0.075926
Y4	1.738.620
Y5	1.270.219
Y6	2.838.418
Y7	1.962.022
Y8	1.558.337
Y9	3.408.169
Y10	1.826.997
Y11	1.623.924
Y12	3.013.956
Y13	1.215.999
Y14	0.768398
Y15	0.731082
Y16	1.372.888
Y17	0.060765
Y18	0.641484
Y19	1.955.399
Y20	0.616721
Y21	0.000000
Y22	0.831788
Y23	1.444.190
Y24	1.241.121
Y25	0.273356
Y26	1.597.101
Y27	1.010.154
Y28	0.152665
Y29	0.966099
Y30	0.975073
Y31	0.000000
Y32	1.178.152

X1	0.000000
X2	0.540467
X3	0.187503
X4	0.272030
Risk	42,94

Table 5: Solution set of the equation for 1 percent expected return given by LINDO Mathematical Optimization Software.

In order to determine a pattern between risk and return, equation is solved for other portfolio return values 0.5, 1.5, 2.0, 2.5 percent by changing the value of p for the following part of the equation.

2.8 X1 +0.4 X2 + 0.5 X3 + 2.9 X4 >= p

For 0.5 % return: 2.8 X1 +0.4 X2 + 0.5 X3 + 2.9 X4 >= 0.5

For 1.5 % return: 2.8 X1 +0.4 X2 + 0.5 X3 + 2.9 X4 >= 1.5

For 2.0 % return: 2.8 X1 +0.4 X2 + 0.5 X3 + 2.9 X4 >= 2.0

For 2.5 % return: 2.8 X1 +0.4 X2 + 0.5 X3 + 2.9 X4 >= 2.5

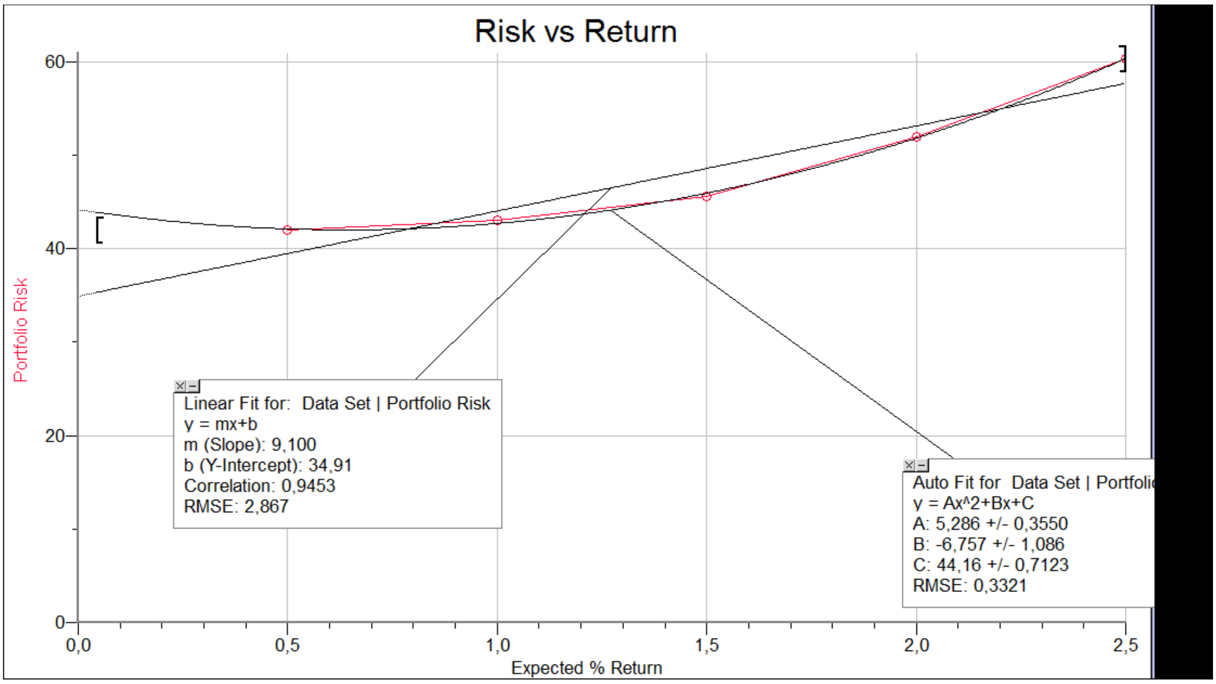
For each percentile value of expected return, the model is solved by LINDO Mathematical Software. From solution sets of those equations, risk values are calculated by the software.

% Return	Risk
0,5	42
1,0	43
1,5	45,6
2,0	51,9
2,5	60,3

Table 6: Portfolio risk calculated for given expected rate of return

6. Conclusion and Evaluation

When risk values found by the optimization software are compared, presence of a pattern can easily be seen. One can say that the old proverbs and quotes about risk taking can be proven because when a Konno Yamazaki Model is used a relation between expected return and portfolio risk is visible.



Graph 1: Relation between portfolio risk and expected percentage rate of return

According to Graph 1, risk is directly proportional to expected return, meaning that if an investor wants to make more money, he/she has to undertake a greater risk. Thus, the research question of this essay; “Is it possible to determine a correlation between financial risk and expected return in economics by using linear portfolio optimization?” can be answered positively.

However, the positive answer can only be given for a specific context. All optimization models require previously recorded data to be used as domains of their functions and a specific time period to be worked on. In addition, for a model to give expected results universe must only consist of included variables. For example, when we take a closer look at the Turkish Stock Market which is the universe of this essay, we can see that the market in fact has 110 basic shares to invest and prices’ of those shares are increasing or decreasing in small periods. Plus, share prices can dramatically go up or down due to significant events going on in the world. For instance, death of an important person, a big merger, acquisition of a company by a rival one or an international crisis may cause the stock market to move like a roller coaster. Nevertheless, a model can not calculate those unexpected incidents because unexpected incidents and their effects' on shares can not be explained mathematically. Hence, a model can only be true for the past with strict limitations and not applicable to real world on a full scale. Although, based on true prices and a correct equation portfolios created by this model are unavailable for commercial use because they are for the past. Furthermore, in real life 0,000000001% risk is dangerous enough to make a billionaire to live in homeless shelter.

Beside from factor mentioned above trustworthiness of a model is also questioned because of assumptions that are made for the model to work[8]. To start with, at the first assumption it was stated that investors are only interested in a single problem which is the optimization problem. However, in real life an investor can be interested in more than one problem causing him to optimize his portfolio differently. Secondly asset returns are assumed as normally distributed random variables. However, statistical data proves that in real markets data is not normally distributed, instead large changes in asset prices are often witnessed. Thirdly, it was assumed that assets have a fixed correlation between them. In contrary, assets have different correlations between each other. Only time assets have fixed correlations between each other is at times of great depressions such as wars and general market crashes. However, at these times, reason for a positive correlation is downwards movement of all assets and this makes MPT especially vulnerable at crisis times. Another assumption was that all investors seek for maximum profit, yet there are investors who would look for assets with less profit for other reasons. Furthermore, it was assumed that all investors are rational and risk averse. However, there are some investors who would go after smaller returns because of smaller risks and behavioral economics state that investors are not rational. Assumption about availability of information to all investors is also incorrect because investors with better sources have more information about the market than smaller investors. In addition investors can't always predict expected return of an asset and sometimes their predictions in fact changes asset prices. Assumption about transaction fees is also incorrect because except very rare scenarios like the given one, an investor always loses more or less money than others because of taxes and other fees. Moreover, market activity always affects stock prices. In contrary to the assumption large transactions and high demand of would cause an asset's price to rise and the opposite

[8] See chapter 3.

would cause the prices to fall down. Additionally, investors have a limited credit to trade in markets. Finally, not all assets are divided into wanted ratios. Assets have certain limits and can't be transacted under the minimum value.

However, as Warren Buffet said simple behavior is more effective. And no matter how unhealthy using a mathematical modem may seem, with its simplicity a model allows us to scientifically correlate what successful businessmen have been saying for years. Despite its disconnection from the real world, a mathematical model is a trustworthy enough way to prove that greater risk means greater profit and prove the hypothesis of this essay.

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Appendix – I

Solution sets for other equations:

- For 0.5% return:

VARIABLE	VALUE
Y1	0.134841
Y2	6.459.194
Y3	0.075926
Y4	1.738.620
Y5	1.270.219
Y6	2.838.418
Y7	1.962.022
Y8	1.558.337
Y9	3.408.169
Y10	1.826.997
Y11	1.623.924
Y12	3.013.956
Y13	1.215.999
Y14	0.768398
Y15	0.731082
Y16	1.372.888

Y17	0.060765
Y18	0.641484
Y19	1.955.399
Y20	0.616721
Y21	0.000000
Y22	0.831788
Y23	1.444.190
Y24	1.241.121
Y25	0.273356
Y26	1.597.101
Y27	1.010.154
Y28	0.152665
Y29	0.966099
Y30	0.975073
Y31	0.000000
Y32	1.178.152
X1	0.000000
X2	0,540467
X3	0,187503
X4	0,27203
Risk	43

Table 7: Solution set of the equation for 0.5 percent expected return given by LINDO Mathematical Optimization Software.

- For 1.5% return:

VARIABLE	VALUE
Y1	0,792688
Y2	6.585.550
Y3	0
Y4	1.411.235
Y5	1.494.229
Y6	3.101.038
Y7	1.924.027
Y8	1.952.873
Y9	3.522.714
Y10	2.451.426
Y11	1.805.769
Y12	3.429.724
Y13	1.930.679
Y14	0,677328
Y15	0,242888
Y16	1.125.940
Y17	0,329527

Y18	1.200.755
Y19	1.965.445
Y20	0
Y21	0,343331
Y22	0,870661
Y23	0,845393
Y24	0,470949
Y25	0,837017
Y26	1.745.622
Y27	0,549101
Y28	0,242051
Y29	1.515.887
Y30	0,821966
Y31	0,423566
Y32	1.077.169
X1	0,12295
X2	0,410669
X3	0,15043
X4	0,31595
Risk	45,6

Table 8: Solution set of the equation for 1.5 percent expected return given by LINDO Mathematical Optimization Software.

- For 2.0% return:

VARIABLE	VALUE
Y1	1.936.164
Y2	6.141.467
Y3	1.300.541
Y4	0.756008
Y5	1.828.110
Y6	2.279.049
Y7	1.823.718
Y8	2.580.375
Y9	2.919.041
Y10	3.315.396
Y11	1.495.900
Y12	4.517.173
Y13	3.125.876
Y14	0.387538
Y15	0.304870
Y16	0.958314
Y17	1.010.145

Y18	1.782.402
Y19	1.862.597
Y20	0.000000
Y21	1.969.500
Y22	0.930862
Y23	0.139582
Y24	0.448010
Y25	0.820172
Y26	1.293.489
Y27	0.000000
Y28	0.284392
Y29	1.814.674
Y30	0.898291
Y31	2.715.175
Y32	0.265738
X1	0.327376
X2	0.102421
X3	0.254671
X4	0.315532
Risk	51,9

Table 9: Solution set of the equation for 2.0 percent expected return given by LINDO Mathematical Optimization Software.

- For 2.5% return:

VARIABLE	VALUE
Y1	3.703.620
Y2	5.604.539
Y3	2.415.794
Y4	0.630456
Y5	1.669.731
Y6	1.739.503
Y7	1.917.612
Y8	2.879.688
Y9	2.848.996
Y10	4.105.346
Y11	1.676.229
Y12	5.065.060
Y13	3.521.672
Y14	0.328682
Y15	0.035684
Y16	0.321040

Y17	1.004.197
Y18	3.099.436
Y19	2.035.601
Y20	0.913316
Y21	1.354.203
Y22	0.724305
Y23	0.000000
Y24	1.471.497
Y25	1.414.361
Y26	1.430.950
Y27	0.086737
Y28	0.404076
Y29	2.192.932
Y30	0.987825
Y31	4.484.706
Y32	0.286095
X1	0.522329
X2	0.000000
X3	0.144903
X4	0.332768
Risk	60.3

Table 10: Solution set of the equation for 2.5 percent expected return given by LINDO Mathematical Optimization Software.