International Baccalaureate Math Analysis and Approaches Higher Level Extended Essay

"To what extent can transformation matrices minimize geometric distortion of a figure's volume when projecting 3D objects onto a 2D plane?"

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Table of Contents

1. Introduction

I started thinking about how 3D objects are represented on a flat surface. I noticed that something always goes wrong and properties such as lengths, areas and angles change along with depth. This led me to ask the question of "How accurately can a 3D object be represented on a 2D surface?" Instead of seeing this from an artistic view, I want to examine this question with mathematics and see what changes on shapes during projection.

In my research, I learned that there are different projection methods that can be used for transformation matrices and algebraic formulas. Both methods has distortion but in different ways. Orthographic and perspective projections are ways to map 3D onto a surface. I will measure this distortion using stuff like length ratios and area scaling. I will also look at if this can be reduced by optimizing certain factors. I plan to study how factors like the viewing distance and how much mathematics can help the preciseness of the projection.

All of this led to my research question: "To what extent can transformation matrices minimize geometric distortion when projecting 3D objects onto a 2D plane?" I want to learn how projection methods can help decrease the distortion that occurs. I plan to use geometry and algebra to model these projection methods with transformation matrices. I also want to compare how much these methods change properties such as lengths and areas etc.

My interest in this topic began in class (and since childhood) with 3D shapes. I realized that others also struggled with this so I decided to learn the background of what is happening and explore it using mathematics. Hopefully, my geometric understanding will improve and I can show how mathematics can help solve a common learning challenge.

2. Background information

2.1. Coordinate model of the object

A point in three dimensional space is represented by the vector: $p=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Lets say we have a unit cube (each side having 1 unit length) lined up with the coordinate axes. Its corners are the set of $V=\{(x, y, z)|\ x, y, z \in \{0,1\}\}$.

This representation allows every geometric thing of the cube to be described using vectors and distance formulas.

For example the length of an edge connecting two vertices $p_1=(x_1,y_1,z_1)$ and $p_2=(x_2,y_2,z_2)$ is given by the Euclidean norm

$$L=\|p_2-p_1\|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}.$$

This formula serves as the initial point for comparing original and projected lengths.

2.2. Rotation transformation in R^3

A rotation transformation is a linear transformation of $T: R^3 \to R^3$ can be represented by a 3×3 matrix R such that

$$T(k) = R_k \text{, k is rotation axis}$$

When a point is rotated about the *z*-axis by angle θ, *z* coordinates remains unchanged. Therefore rhe transformation can be derived by analysing the rotation in the xy plane.

Let p is a point that p(x,y,z) represented by the vector $p=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

Let $r = \sqrt{x^2 + y^2}$ be the distance of the point from the origin in the xy plane and α be the angle between the point and the positive x-axis. So, the polar coordinates of p:

$$x = r\cos\alpha$$

$$y = r\cos\alpha$$

After rotating the point by θ, the new angle becomes θ+α, so the new polar coordinates of point are:

$$x' = rcos(\theta + \alpha)rcos\theta cos\alpha - rsin\theta sin\alpha = xcos\theta - ysin\theta$$

$$y\ = rsin(\theta + \alpha) = \mathrm{rsin}\theta cos\alpha + rsin\alpha cos\theta = xsin\theta + ycos\theta$$

$$z = z$$

Rotated point is called p_r that:

$$p_r = (\mathrm{xcos\theta - sin\theta, xsin\theta + ycos\theta, z}) = \begin{pmatrix} \mathrm{cos\theta} & -\mathrm{sin\theta} & 0 \\ \mathrm{sin\theta} & \mathrm{cos\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{x} \\ \mathrm{y} \\ \mathrm{z} \end{pmatrix} \text{ where } \mathrm{R_z(\theta)} = \begin{pmatrix} \mathrm{cos\theta} & -\mathrm{sin\theta} & 0 \\ \mathrm{sin\theta} & \mathrm{cos\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This matrix represents a rotation about the *z*-axis by angle *θ*. Applying this matrix changes the coordinates of every vertex while keeping the distances in R^3 since rotations are isometries.

Similarly the matrix which represents the rotation about the y-axis by angle θ is

$$\mathrm{R_y(\theta)} = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

2.3. Orthographic projection as a linear map

Orthographic projection onto the *xy* plane is a linear transformation of $P_o: R^3 \rightarrow R^2$ defined by $P_o(x, y, z) = (x, y)$. In projection matrix form $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is transformed to $P_o = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

Since this transformation ignores the z component it is not injective (different points in R^3 with the same (*x, y*) coordinates are mapped to the same point in R^2). This loss of information is one of the basic mathematical sources of distortion in orthographic projection.

2.4. Perspective projection as a non linear mapping

Perspective projection is a non linear transformation of $P_p: R^3 \rightarrow R^2$ defined by mapping $P_p(x,y,z) = (\frac{x}{z+d}, \frac{y}{z+d})$ where d>0 is the distance between the observer and the projection plane. When a point (x, y, z) is projected onto the plane, the projected coordinates are scaled according to z. Using similar triangles the scaling factor becomes $\frac{1}{z+d}$(this is because the perspective projection is non linear). When you increase d the depth effect is lowered and the transformation becomes more like an orthographic projection. If you decrease d the perspective distortion becomes stronger. That is why objects that are farther away look smaller in a perspective projection.

2.5. Mathematical definition of distortion

Distortion is defined as the deviation of geometric quantities under a projection mapping. For an edge represented by two points $p_1,p_2 \in R^3$ the original length is

$$L=\|p_2-p_1\|.$$

After projection the related points are $p_1,p_2 \in R^2$, and the projected length is

$$L'=\|p'_2-p'_1\|.$$

The length distortion ratio is then defined as

$$D_L = \frac{L'}{L} \text{ .}$$

A value of $D_L = 1$ shows perfect preservation of length while values different from 1 shows how much distortion there is

3. Applying the Model - 1 (Orthographic Projection of a Rectangular Prism)

3.1. Defining the object

The rectangular prism has the dimensions of $2\times1\times1$ and is placed so that one corner is at the origin and its edges are lined up with the coordinate axes.

The eight vertices of the prism are: A(0,0,0), B(2,0,0), C(2,1,0), D(0,1,0), E(0,0,1), F(2,0,1), G(2,1,1), H(0,1,1). This model allows each edge and face of the prism to be analysed algebraically under projection.

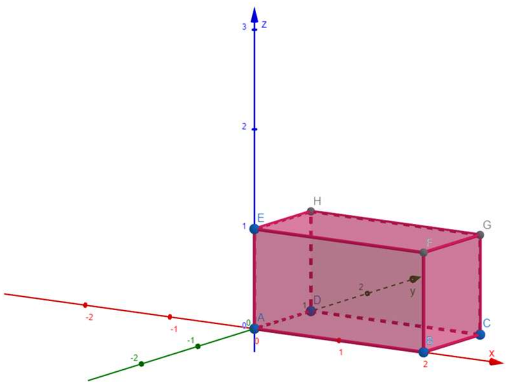

Figure 1: Rectangular Prism

3.2. Applying orthographic projection

Orthographic projection onto *xy* plane is defined by $P_o(x,y,z) = (x,y)$.

Applying this transformation to each gives:

Vertex	2D Orthographic Projection
A(0,0,0)	(0,0)
B(2,0,0)	(2,0)
C(2,1,0)	(2,1)
D(0,1,0)	(0,1)
E(0,0,1)	(0,0)
F(2,0,1)	(2,0)
G(2,1,1)	(2,1)
H(0,1,1)	(0,1)

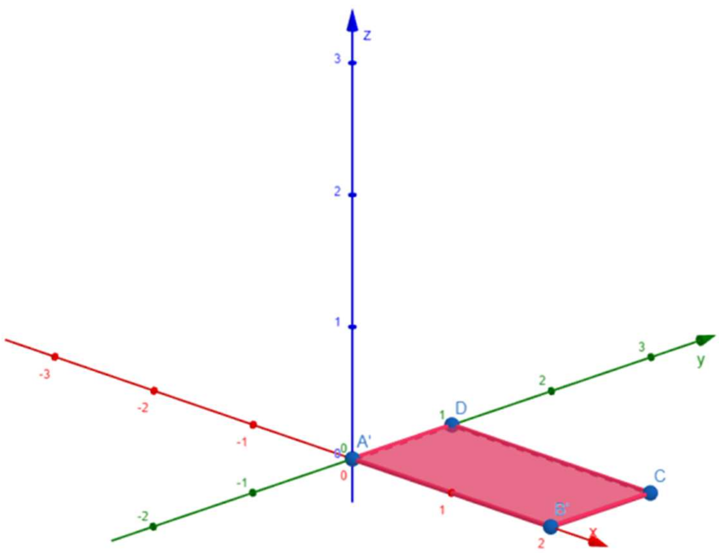

Figure 2: 2D Projection

As shown above all pairs of vertices that only differ in their z coordinate end up at the same point. This shows that orthographic projection removes all depth information.

3.3. Analysing length distortion

3.3.1. Original edge lengths

- Edges parallel to the *x*-axis have length $L_x=2$
- Edges parallel to the *x*-axis have length $L_y=1$
- Edges parallel to the *x*-axis have length $L_z=1$

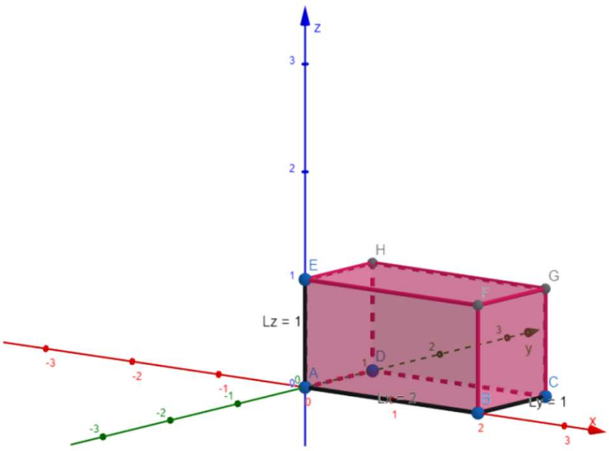

Figure 3: Original Edge Lengths

3.3.2. Projected edge lengths

- Edge AB (along the *x*-axis)

A(0,0,0), B(2,0,0)

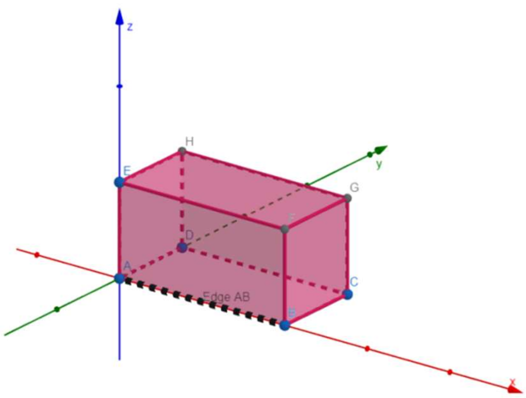

Figure 4: Edge AB

After projection: A'(0,0), B'(2,0)

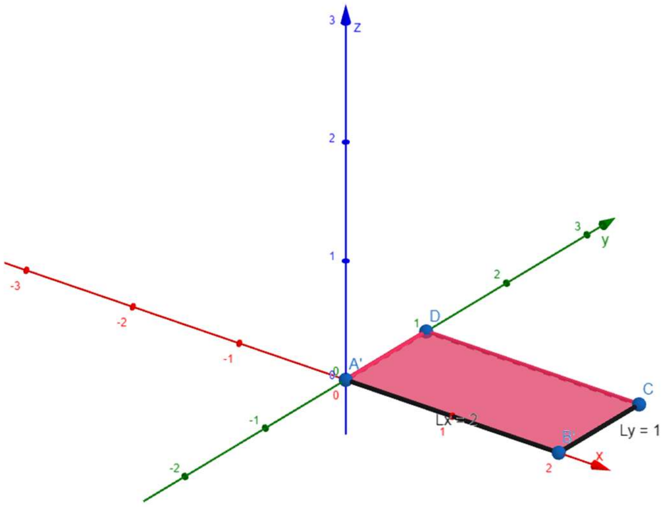

Figure 5: After Projection

Projected length is $L'_{AB}=\sqrt{(2-0)^2+(0-0)^2}=2$

Distortion ratio is $D_L(\mathrm{AB})=\frac{L'_{\mathrm{AB}}}{L_{\mathrm{x}}}=\frac{2}{2}=1.$

- Edge AD (along the *y*-axis)

A(0,0,0), D(0,1,0)

After projection: A'(0,0), D'(0,1)

Projected length is $L'_{AD}=1.$

Distortion ratio is $D_L(AD)=\frac{1}{1}=1$

- Edge AE (along *z*-axis)

A(0,0,0), E(0,0,1)

After projection: A'(0,0), E'(0,0)

Projected length is $L'_{AE}=0.$

Distortion ratio: $D_L(AD)=\frac{0}{1}=0.$

3.4. Interpreting the results

These results show that orthographic projection keeps all lengths that lie parallel to the projection plane while completely ignoring lengths in the direction perpendicular to the plane. This shows us a important observation that orthographic projection distortion depends completely on the position of an edge relative to the projection plane and not on its original length. Edges parallel to the xy-plane keeps their length (regardless of whether they are long or short) while edges perpendicular to the plane lose all length information. This limit supports the use of perspective projection which takes depth into account and therefore might reduce the distortion for edges lined up with the z direction.

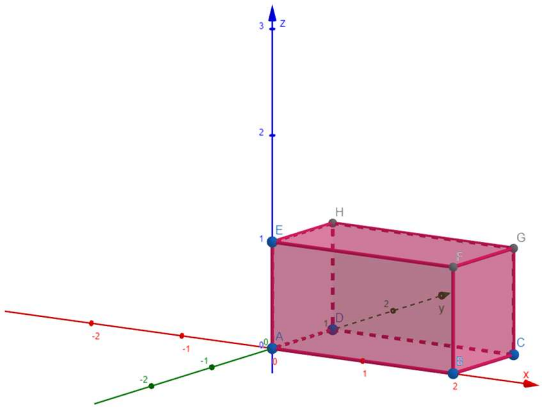

Figure 6: Before Orthographic Projection

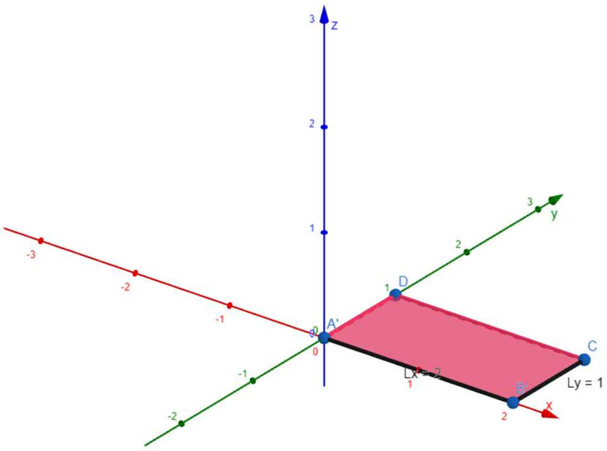

Figure 7: After Orthographic Projection

4. Applying the Model - 2 (Perspective Projection of a Rectangular Prism)

4.1. Defining the projection model

In this investigation perspective projection is modelled by the transformation

$$P_p(x, y, z) = \left(\frac{d \cdot x}{z + d}, \frac{d \cdot y}{z + d}\right),$$

where d > 0 represents the distance between the observer (the person who is looking at the object) and the projection plane. Multiplying by d ensures that if a point lies exactly on the projection plane (z=0), the projection is 1:1. For example if z=0, the formula becomes $\frac{d \cdot x}{0+d}=x$. The division by $z + d$ is the thing that makes "distance makes things smaller". As the object's z value increases, the denominator $(z + d)$ gets bigger which makes the resulting x and y coordinates smaller. Not like the orthographic projection this mapping relies on the z-coordinate. As a result the points that are further away from the observer appear smaller which shows us a different type of distortion.

4.2. Applying perspective projection to the vertices

The rectangular prism has the same vertices as before

A(0,0,0), B(2,0,0),C(2,1,0),D(0,1,0),
E(0,0,1),F(2,0,1),G(2,1,1),H(0,1,1).

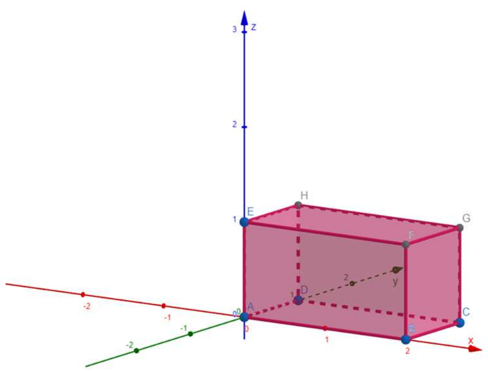

Figure 8: Rectangular Prism

Applying the perspective projection gives:

- **Vertices with z = 0**

  For points on the plane $z = 0$, $x' = \frac{d \cdot x}{0+d} = x$ and $y' = \frac{d \cdot y}{0+d} = y$

  So, A'(0,0), B'(2,0), C'(2,1), D'(0,1).

- **Vertices with z = 1**

  For points with z = 0, $x' = \frac{d \cdot x}{1+d}$ and $y' = \frac{d \cdot y}{1+d}$

  So, E'(0, 0), F'($\frac{2d}{d+1}$,0), G($\frac{2d}{d+1}$, $\frac{d}{d+1}$), H'(0,$\frac{d}{d+1}$).

Vertex	2D Perspective Projection
A(0,0,0)	A'(0,0)
B(2,0,0)	B'(2,0)
C(2,1,0)	C'(2,1)
D(0,1,0)	D'(0,1)
E(0,0,1)	E'(0,0)
F(2,0,1)	F'($\frac{2d}{d+1}$,0)
G(2,1,1)	G'($\frac{2d}{d+1}$, $\frac{d}{d+1}$)
H(0,1,1)	H'($\frac{2d}{d+1}$, $\frac{d}{d+1}$)

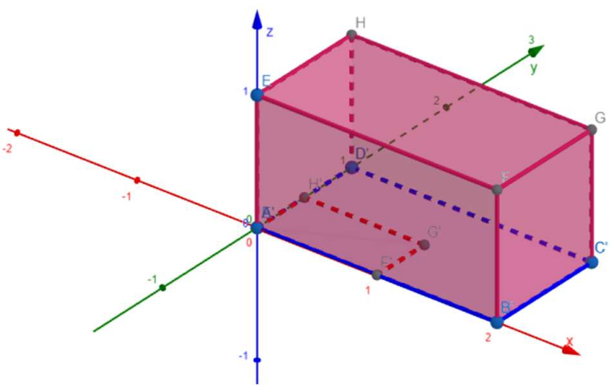

Figure 9: When d =1 (E.g. For a small value of d, high distortion)

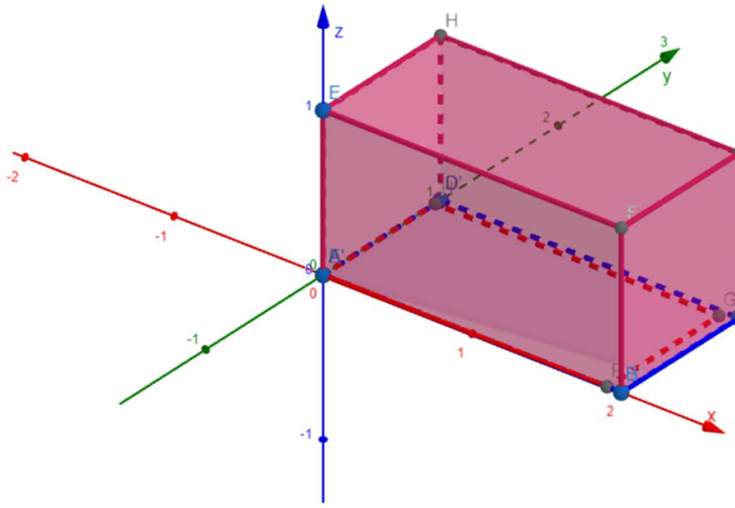

Figure 10: When d =18 (E.g. For a large value of d, low distortion)

Since, for $z > 0$, $d<d+z$ then $\frac{d}{d+z}<1$. This proves mathematically that as z increases, the image dimensions shrink relative to the original object.

4.3. Analysing length distortion

- Edge AB (along the x-axis, z=0)

Original length: $L_{AB}=2$.

Projected endpoints: A'(0,0), B'(2,0).

Projected length: $L'_{AB}=2\text{-}0=2$.

Length distortion ratio: $D_L(AB)=\frac{L'_{AB}}{L_{AB}}=\frac{2}{2}=1$.

- Edge EF (along the x-axis, z = 1)

Original length: $L_{EF}=2$.

Projected endpoints: $E'(0, 0)$, $F'(\frac{2d}{d+1}, 0)$.

Projected length: $L'_{EF}=\frac{2d}{d+1}$.

Length distortion ratio: $D_L(EF)=\frac{\frac{2d}{d+1}}{2}=\frac{d}{d+1}$.

- Edge AE (along the z-axis)

  Original length: $L_{AE}=1$.

  Projected endpoints: $A'(0,0)$, $E'(0,0)$.

  Projected length: $L'_{AE}=0$.

  Length distortion ratio: $D_L(EF)=\frac{L'_{EF}}{L_{EF}}=\frac{0}{1}=0$.

This shows that even perspective projection can not preserve edges parallel to the z-axis since both end points project to the same point in the plane.

4.4. Comparing orthographic and perspective distortion

- Under orthographic projection:
  - D_L(AB)= 1
  - D_L(AD)= 1
  - D_L(AE)=0
- Under perspective projection:
  - D_L(AB)= 1
  - D_L(EF)= $\frac{d}{d+1}$
  - D_L(AE)= 0

This comparison shows that while orthographic projection keeps all lengths in the plane perfectly, perspective projection also keeps lengths on the specific plane where $z=0$. While some distortion is unavoidable, perspective projection allows it to be adjusted mathematically which creates an opportunity for optimisation.

4.5 Defining a distortion function for perspective projection

To study this I defined a function for distortion in which it is based on length preservation for edges parallel to the x-axis. The function measures the total deviation of the projected edge lengths from their original values. Since front face (z=0): $D_L(\text{AB})=1$ and rear face ($z=1$)=$D_L(\text{EF})=\frac{d}{d+1}$, let

$$\mathrm{D}(d)=|1-1|+|1-\frac{d}{d+1}|$$

Which can be simplified to

$$\mathrm{D}(d)=\left|1-\frac{d}{d+1}\right|$$

This function measures the total deviation of the projected edge lengths from their original values for edges at different depths. By calculating the sum of the absolute differences between the original lengths and the projected lengths, we can measure the difference between the original lengths and the projected lengths. The goal is now to find the value of d that minimises $\mathrm{D}(d)$.

4.6. Optimising the distortion

To simplify the analysis I first removed the absolute values by noting that for d > 0,

$$\frac{d}{d+1}<1.$$

So,

$$\mathrm{D}(d)=(1-1)+\left(1-\frac{d}{d+1}\right)=1-\frac{d}{d+1}.$$

To minimise D(d), I differentiate with respect to d;

$$\mathrm{D}'(d)=-\frac{(d+1)-d}{(d+1)^2}=-\frac{1}{(d+1)^2}$$

Since

$$D'(d)<0 \quad \text{for all } d>0,$$

the function $D(d)$ is strictly decreasing on $d>0$. This means that distortion is minimised when d increases. Meanwhile a very large value of d minimizes distortion, it also weakens the perspective effect that provides a sense of depth and immersion. This leads to an important result that there is a exchange between mathematical minimisation and practical limits.

4.7. Interpretation of the optimisation

Increasing d reduces distortion for edges parallel to the x-axis. As d goes towards infinity;

$$\lim_{d\to\infty}\frac{d}{z+d}=\frac{\infty}{\infty}$$

So, we can use L'Hospital rule. Then,

$$\lim_{d\to\infty}\frac{(d)'}{(z+d)'}=\lim_{d\to\infty}\frac{1}{1}=1$$

which means that the projection begins to behave like an orthographic one where lengths are kept perfectly.

In reality applications like very large values of d would cause object to lose its perspective effects. Vicaversa, when d is small, the "scaling" difference between the front (z=0) and the back (z=1) becomes too big and leads to visual instability (how "sensitive" the image is to tiny changes in position).

This shows that "minimising distortion" cannot be understood only in a theoretical sense. Instead it must be considered within a real range of parameters. This understanding shows one of the strengths of mathematical modelling: it does not only provide exact results but also clear up the limitations of those results.

5. Applying the Model - 3 (Effect of rotation on distortion)

In the section before this one distortion was analysed for a rectangular prism lined up with the coordinate axes. But in real life objects are not so often viewed in such a perfectly aligned position. This leads us to an important question that "Can distortion be reduced by changing the position of the object before projection?".

To investigate this I will now explain a rotation as an additional variable in the model.

5.1 Rotating the Prism

I consider a rotation of the rectangular prism about the *z*-axis by an angle θ.

This rotation is represented by the matrix

$$R_z(\theta)=\begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}.$$

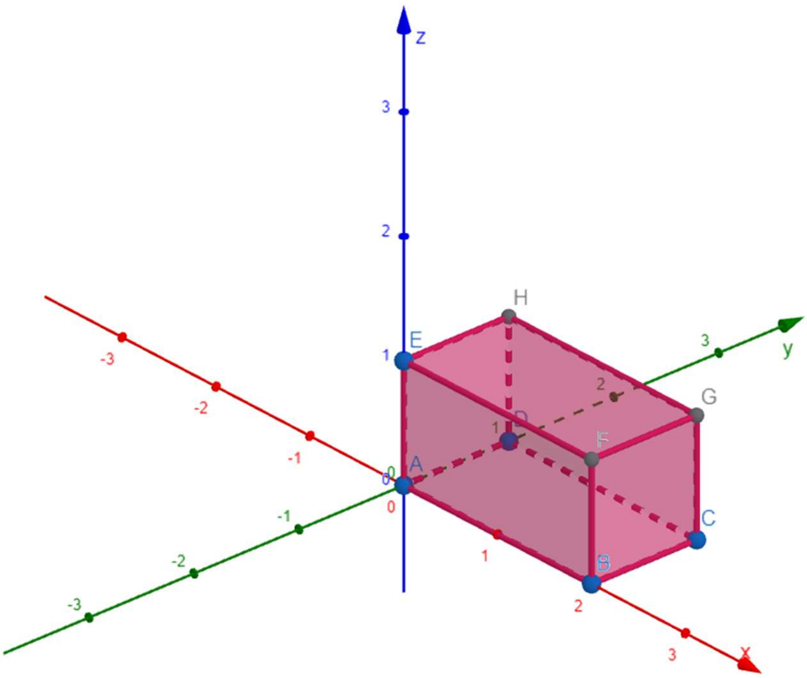

Figure 11: Before Rotation

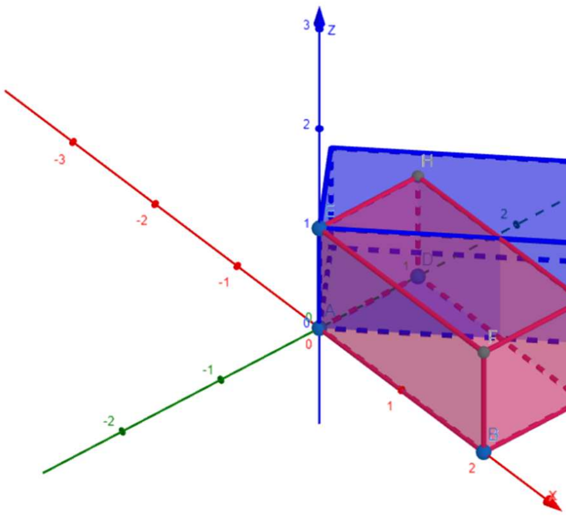

Figure 12: After rotation (blue) (about 45 degrees here)

For any vertex p, the rotated position is

$$p_r = R_z(\theta)p.$$

This transformation keeps all distances in three dimensional so any change in distortion after projection is due only to the projection not to the rotation itself.

For our rectangular prism vertices of the prism after rotation about the z-axis by angle θ are shown below:

Vertex	Rotation
A(0,0,0)	A'(0,0,0)
B(2,0,0)	B'$(2\cos\theta, 2\sin\theta, 0)$
C(2,1,0)	C'$(2\cos\theta - \sin\theta, 2\sin\theta + \cos\theta, 0)$
D(0,1,0)	D'$(-\sin\theta, \cos\theta, 0)$
E(0,0,1)	E'(0,0,1)
F(2,0,1)	F'$(2\cos\theta, 2\sin\theta, 1)$
G(2,1,1)	G'$(2\cos\theta - \sin\theta, 2\sin\theta + \cos\theta, 1)$
H(0,1,1)	H'$(-\sin\theta, \cos\theta, 1)$

5.2. Projection after Rotation

After rotating the prism, I apply perspective projection

$$P_p(x,y,z)=\left(\frac{dx}{z+d},\frac{dy}{z+d}\right).$$

The combined transformation can therefore be written as like

$$p'=\left(\frac{d(x\cos\theta-y\sin\theta)}{z+d},\frac{d(x\sin\theta+y\cos\theta)}{z+d}\right)$$

This shows that the position of final state of each point on the plane relies on two factors:

- the viewing distance d,
- the rotation angle θ.

Vertex	Perspective Projection after Rotation
A’(0,0,0)	A’(0, 0)
B’(2cosθ, 2sinθ, 0)	B’(2cosθ, 2sinθ)
C’(2cosθ − sinθ, 2sinθ + cosθ, 0)	C’(2cosθ − sinθ, 2sinθ + cosθ)
D’(−sinθ, cosθ, 0)	D’(−sinθ, cosθ)
E’(0,0,1)	E’(0,0)
F’(2cosθ,2sinθ,1)	F’$\left(\frac{2d\cos\theta}{1+d},\frac{2d\sin\theta}{1+d}\right)$
G’(2cosθ−sinθ,2sinθ+cosθ,1)	G’$\left(\frac{2d\cos\theta-\sin\theta}{1+d},\frac{d(2\sin\theta+\cos\theta)}{1+d}\right)$
H’(−sinθ,cosθ,1)	H’$\left(\frac{-d\sin\theta}{1+d},\frac{d\cos\theta}{1+d}\right)$

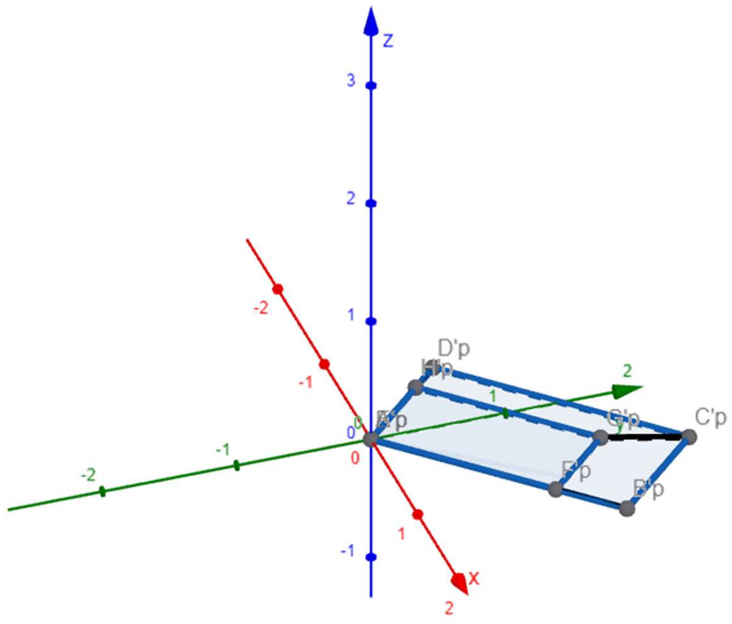

Figure 13: Perspective projected points after rotations

5.3. Distortion as a function of θ

To study how position affects distortion I focus on length distortion for edges normally parallel to the x-axis. Before rotation an edge vector is

$$\mathrm{v} = (2,0,0).$$

After rotation it is

$$\mathrm{v_r}=\mathrm{R_z}(\theta)\mathrm{v}=(2\cos\theta,\ 2\sin\theta,\ 0).$$

This shows that rotation repositions the original x-direction length between the x and y directions in the plane.

After perspective projection (for edges with z = 0) the projected vector becomes

$$\mathrm{v_r}'=(2\cos\theta, 2\sin\theta)$$

Its projected length is therefore

$$\mathrm{L}'(\theta)=\sqrt{(2\cos\theta)^2+(2\sin\theta)^2}=2$$

Since $L'(\theta)=2$ and the original length was 2, rotation in the plane $z=0$ has zero distortion.

5.4. Rotation and depth dependent distortion

Consider now an edge parallel to the x-axis at depth $z = 1$.

Before rotation the edge vector is again (2,0,0).

After rotation it becomes

$$v_r=(2\cos\theta, 2\sin\theta, 0).$$

But its position vector has z = 1 so perspective projection gives

$$v_r'=\left(\frac{2d\cos\theta}{d+1}, \frac{2d\sin\theta}{d+1}\right)$$

The projected length is

$$L'(\theta)=\frac{2d}{d+1}$$

Once again the projected length $L'(\theta)=\frac{2d}{d+1}$ does not depend on θ. However, because $\frac{2d}{d+1}<2$ for all $d>0$, this shows that edges at depth $z=1$ are consistently shrunken regardless of their rotation about the z-axis. This result is important because it confirms that rotation about the z-axis alone is not enough to eliminate depth dependent length distortion. It shows a limit of a simple optimization that not every factor change (like θ) can counteract the inherent distortion caused by the viewing distance d.

5.5 Rotating about a different axis

To achieve a more meaningful reduction in distortion I will consider rotation about the y-axis which changes how much of each edge lies in the direction of depth.The rotation matrix about the y-axis is

$$R_y(\theta)=\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}.$$

Applying this to an edge originally along the x-axis gives

$$v_r = (2\cos\theta, 0, -2\sin\theta).$$

Now the edge has a non zero z-component. Because the depth varies across the edge, the perspective projection formula $P_p(x,y,z) = \left(\frac{dx}{z+d}, \frac{dy}{z+d}\right)$ will affect different parts of the edge differently. Specifically, since the z-coordinate is no longer a constant for all points on the edge, the projection is no longer a simple linear scaling of the original length.

5.6. Distortion function with rotation

To simplify the results I will focus on the midpoint of the edge and use its depth to estimate the scaling factor. The midpoint of the rotated edge has depth

$$z_m = 1 - \sin\theta.$$

The projected length is estimated

$$L'(\theta) \approx \frac{2d\cos\theta}{d + z_m} = \frac{2d\cos\theta}{d + 1 - \sin\theta}.$$

The original length is L = 2 so the distortion ratio is

$$D_L(\theta) = \frac{L'(\theta)}{L} = \frac{d\cos\theta}{d + 1 - \sin\theta}.$$

This gives a distortion function depending only on θ. This model shows that as d increases the ratio approaches $\cos\theta$ which is consistent with the behavior of an orthographic projection.

5.7 Optimising with respect to θ

To minimise distortion, I differentiate $D_L(\theta)$:

$$D_L(\theta)=\frac{d\cos\theta}{d+1-\sin\theta}.$$

Using the quotient rule,

$$D_L{}'(\theta)=\frac{d(-\sin\theta)(d+1-\sin\theta)-(d\cos\theta)(-\cos\theta)}{(d+1-\sin\theta)^2}$$

Simplifying the numerator:

$$N(\theta)=-d(d+1)\sin\theta+d\sin^2\theta+d\cos^2\theta$$

$$N(\theta)=-d(d+1)\sin\theta+d(1)$$

$$N(\theta)=d[1-(d+1)\sin\theta]$$

So the derivative is,

$$D_L{}'(\theta)=\frac{d[1-(d+1)\sin\theta]}{(d+1-\sin\theta)^2}$$

Setting $D_L'(\theta)=0$ to find the stationary point:

$$1-(d+1)\sin\theta=0\Rightarrow\sin\theta=\frac{1}{d+1}.$$

This result shows that distortion is minimised (we can test it from second derivative test, where $D_L''(\theta)>0$) when the rotation angle satisfies

$$\sin\theta=\frac{1}{d+1}.$$

5.8 Interpretation

This is a key result of the investigation. It shows that:

- Distortion cannot be analyzed by looking at a single variable alone. The best position (θ) is only dependent on the viewing distance (d).

By combining perspective projection with an appropriate rotation, it is possible to find a specific point that minimizes length distortion for edges at depth. As the observer distance d increases the needed "correction" angle θ decreases, as the projection approaches orthographic behavior.

This supports the main idea of the essay that transformation matrices can be used not only to describe distortion, but also to control and minimise it.

6. Evaluation

Overall, throughout this writing process, I saw how a 3D object could be represented on a 2D plane while minimizing distortion. By comparing the two projections and then optimizing through rotation, I was able to numerically analyze this distortion as well.

The most surprising aspect of this thesis for me was observing how different projection methods are from each other. Orthographic projection left only the lengths in directions parallel to the projection plane as they were but ignored depth. While this was easy and practical for drawing, it caused the edges to appear very distorted. Perspective projection also has some distortion, but the distortion is much less than orthographic projection because it scales objects according to how far away they are from the viewer, essentially proportioning them. So in the end I learned that what looks "correct" doesn't always mean it's geometrically correct.

One of the key points of this study was that distortion was designed not as a general concept, but as a measurable and repeatable process. And so it was, demonstrating that distortion could be minimized by properly orienting the object before projection. Of course, this study has some limitations as it can be more detailed than it is already. First, since the main focus of my models was length distortion, I only addressed a small portion of other important geometric aspects. A more detailed study could actually create a distortion measure that includes all three. Second, I based the optimization on some simplifying assumptions, such as using the midpoint of an edge to determine depth. This made most of the calculations easier, but unfortunately, it didn't fully explain how distortion varies along an edge. Another limitation is that the model is within an ideal mathematical framework (which is almost impossible to achieve in real life), such as assuming a perfect rectangular prism, a fixed projection plane, and a single observer at a single location. Therefore, I believe the results of this research should be viewed theoretically rather than directly as practical rules.

Even with all these limitations, I believe the research achieved its main objective. Starting from my own experience with visualizing three-dimensional objects on paper, I managed to develop a mathematical model that explains why this difficulty exists and how it can be reduced. And most importantly, this process allowed me to see geometry not just as a set of shapes, but as a system that can be analyzed, modeled, and optimized using algebra and calculus.

If this investigation were more detailed I would really want to search the area distortion in a better depth and compare it with length distortion to see if the same optimisation applies. Another thing would be to explore projective geometry which provides a more general perspective and may give deeper theoretical explanations for the results we got here.

Overall this investigation shows that representing a 3D world on a 2D plane will always involve compromise. However, mathematics makes it possible to understand these compromises precisely and to some extent, to manage them. This supports the main point of the article which is that distortion cannot be completely gotten out of but it can be observed, analyzed and greatly reduced by mathematical modeling and optimization.

7. Bibliography

GeeksforGeeks. (2025, July 15). *Perspective projection and its types*. https://www.geeksforgeeks.org/computer-graphics/perspective-projection-and-its-types/

The geometry of perspective projection. (n.d.-c). https://www.cse.unr.edu/~bebis/CS791E/Notes/PerspectiveProjection.pdf

Orthographic projection. (n.d.-c). https://abs.cu.edu.tr/Dokumanlar/2016/EEE114/195782230_eee114_lecture_notes_7._week.pdf

Libretexts. (2025, September 22). *2.2: Orthographic projections*. Engineering LibreTexts. https://eng.libretexts.org/Courses/Illinois_Institute_of_Technology/Introduction_to_Engineering_Drawing_and_Design/02%3A_Module_B_-_Orthographic_projections/2.02%3A_Orthographic_projections

1 3. orthographic projection and multi-view projection. (n.d.-a). https://transport.itu.edu.tr/docs/librariesprovider99/dersnotlari/dersnotlarires112e/not/cadd-3.pdf?sfvrsn=4

3D Calculator. GeoGebra. (n.d.). https://www.geogebra.org/3d