Modelling J2-Driven Nodal and Apsidal Precession in Sun-Synchronous Orbits Using Lagrange Planetary Equations

Mathematics: Analysis & Approaches HL

Extended Essay

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Contents

Introduction

Satellites are used in virtually every area of modern life. The orbits that those satellites follow are explained by Kepler's Laws, which assume that the Earth is a perfect sphere. The Earth, however, is not a perfect sphere; it is slightly flattened at the poles and wider at the equator. These differences in the Earth's shape can have an impact upon the orbits of satellites in space. Thus, certain orbital elements of those satellites may change over time.

This essay will investigate the effect of the Earth's oblateness (represented by the parameter J_2) upon the Right Ascension of the Ascending Node (Ω) and the Argument of Perigee (ω), and how those variables are affected for sun-synchronous orbits. The analysis of real data upon satellite orbits will be performed, specifically for satellites like Landsat 8 and Sentinel-2A. This topic is a popular one to investigate for satellite orbits due to the fact that orbital perturbations are an excellent example of how abstract mathematical tools can be used to accurately model and predict physical phenomena. The Earth's oblateness is the primary factor in the secular perturbations of satellites in sun-synchronous orbits.

Research Question and Essay Structure

Proposed research question:

How can the secular rates of change $\dot{\Omega}$ and $\dot{\omega}$ caused by Earth's oblateness (J_2) be derived from first principles using Lagrange planetary equations?

Major section in this essay	What it achieves
Notation and Foundations	Makes orbital mechanics readable for a non-expert; fixes units + symbols; introduces J_2, TLEs, and orbital elements.
Derivation of $\dot{\Omega}$ and $\dot{\omega}$	Shows every chain-rule step, orbit averaging integral, and simplification to standard forms using $n = \sqrt{\mu/a^3}$.
Numerical evaluation (Landsat 8, Sentinel-2A)	Uses real, cited mission orbits and TLE-derived elements to compute $a, p, n, \dot{\Omega}, \dot{\omega}$ in SI and deg/day.
Sensitivity analysis	Quantifies dependence on a, e, i and provides tables + plots.
Validation and uncertainty	Cross-checks algebra against published forms; introduces TLE empirical drift method; bounds neglected forces and provides error bars.
Conclusion	Summarizes results and limitations in a structured way.

Table 1 summarizes how each section of the essay addresses the research objectives.

Notation and Orbital Mechanics Foundations

Notation Table and Unit Conventions

SI units are used in the text unless stated otherwise. For example, when a parameter is given in kilometers, it is immediately converted using:

$$1 \text{ km} = 1000 \text{ m}.$$

For example, $a = 6870 \text{ km} = 6.870 \times 10^6 \text{ m}$.

Symbol	Meaning	Units used here
a	semi-major axis (size of ellipse)	m
e	eccentricity (shape; 0=circle)	dimensionless
i	inclination (tilt vs Earth's equator)	rad (internally), degrees (reported)
Ω	Right Ascension of Ascending Node (RAAN): angle locating the ascending node on the equator	rad / degrees
ω	argument of perigee: angle from ascending node to perigee (within orbital plane)	rad / degrees
ν	true anomaly: angle from perigee to spacecraft position	rad
$u = \omega + \nu$	argument of latitude	rad
r	satellite distance from Earth center	m
$p = a(1 - e^2)$	semi-latus rectum	m
n	mean motion $= \sqrt{\mu/a^3}$	rad/s
μ	Earth gravitational parameter GM	m^3/s^2
R_E	Earth reference equatorial radius	m
J_2	second zonal harmonic (oblateness)	dimensionless
J_3	third zonal harmonic (pear-shape)	dimensionless

For Earth constants I use values listed in a NASA/JPL NAIF kernel (WGS84 set), including R_E, J_2, and J_3.

What are Ω and ω, Physically?

A Keplerian orbit in 3D is described by six classical orbital elements. Two of these elements describe the orientation of the orbital plane in space:

- The angle Ω (RAAN) describes the angle between the reference direction (the First Point of Aries) and the ascending node, measured eastward within the equatorial plane.
- The angle ω describes the angle between the ascending node line and the perigee (the point in the orbit where the satellite is closest to Earth).

Earth's gravity is not perfectly spherical; it is slightly flattened. That flattening produces a torque-like effect upon the orbit of the satellite, causing that orbital plane to slowly rotate about the Earth's rotation axis. For an SSO, that rate of plane rotation is deliberately set to be close to that of the mean apparent motion of the Sun ($\approx 0.9856°/\text{day}$).

Rationale for Utilizing Mission-Anchored Cases and TLE Data

The satellites selected for modelling in this investigation are Landsat 8 and Sentinel 2A, both of which are classic optical Earth-observation missions that operate in Sun Synchronous Orbits (SSOs). These orbits have well-documented typical parameters:

- Landsat 8 satellites have a near-polar Sun-synchronous orbit, with an inclination of about 98.2 degrees and an altitude of about 705 km.
- Sentinel-2 satellites have a Sun-synchronous orbit with an altitude of 786 km and a period of about 100 minutes.

The Two-Line Elements (TLE) data for both of these satellites is publicly available, and the format of these TLE data lines includes the mean orbital elements that are used with orbital propagation algorithms like SGP4. The format of the TLE data is documented by CelesTrak.

Derivation of $\dot{\Omega}$ and $\dot{\omega}$ from the J_2 Perturbation

From Ideal Kepler Motion to Why J_2 Matters

The gravitational potential of a perfect sphere Earth model is:

$$U_0(r) = -\frac{\mu}{r}.$$

This yields closed conic-section orbits with constant elements $(a, e, i, \Omega, \omega, M)$.

Because the Earth is not perfectly spherical, a standard way to represent its varying gravity field is through a spherical harmonic expansion. Retaining only the dominant non-spherical zonal term yields a perturbing potential proportional to J_2, which is the leading "oblateness" correction. This is also the dominant driver of the secular node precession for LEO sun-synchronous orbits.

The J_2 Disturbing Potential and Converting Latitude to Orbital Geometry

A widely used zonal-harmonic form (per unit mass) can be written:

$$U(r, \phi) = -\frac{\mu}{r}\left(1 - J_2\left(\frac{R_E}{r}\right)^2 P_2(\sin\phi) + \cdots\right),$$

where ϕ is geocentric latitude and P_2 is the second Legendre polynomial:

$$P_2(x) = \frac{1}{2}(3x^2 - 1).$$

The disturbing potential (also called the disturbing function in many celestial mechanics conventions) is the part beyond the central term:

$$\mathcal{R}_{J_2} = U - U_0 = -\frac{\mu}{r}\left(1 - J_2\left(\frac{R_E}{r}\right)^2 P_2(\sin\phi)\right) - \left(-\frac{\mu}{r}\right) = \frac{\mu}{r}J_2\left(\frac{R_E}{r}\right)^2 P_2(\sin\phi).$$

Substituting P_2:

$$\mathcal{R}_{J_2} = \frac{\mu J_2 R_E^2}{r^3} \cdot \frac{1}{2}(3\sin^2\phi - 1) = \frac{\mu J_2 R_E^2}{2r^3}(3\sin^2\phi - 1).$$

To perform the orbit-averaging integral, $\sin\phi$ must be expressed in terms of the classical orbital elements. In an Earth-centered inertial frame where the z-axis is Earth's spin axis, the orbital-plane geometry gives (this is stated explicitly in Kozai's classic derivation):

$$\sin\phi = \sin i \sin(\nu + \omega).$$

Kozai writes the same relation in his notation: $\sin\delta = \sin i \sin(v + \omega)$.

Defining the argument of latitude:

$$u = \omega + \nu.$$

Then

$$\sin^2\phi = \sin^2 i \sin^2 u.$$

Substituting into $\mathcal{R}_{J_2}$:

$$\mathcal{R}_{J_2} = \frac{\mu J_2 R_E^2}{2r^3}(3\sin^2 i \sin^2 u - 1).$$

Preparing for Orbit Averaging

Secular rates in the Lagrange method come from using the orbit-averaged disturbing function (average over one orbital period):

$$\overline{\mathcal{R}} = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{R}\ dM,$$

where M is mean anomaly.

To compute this average, the relationships between M, ν, and r must be established. For a Keplerian ellipse:

$$r = \frac{p}{1 + e\cos\nu}, \quad p = a(1 - e^2).$$

Also, converting the integral from M to ν requires dM in terms of $d\nu$. Kozai explicitly uses a standard identity (attributed to classical celestial mechanics formulae) of the form:

$$dM = (1 - e^2)^{3/2} \frac{d\nu}{(1 + e\cos\nu)^2}$$

Orbit-Averaging Integral

Starting with:

$$\overline{\mathcal{R}}_{J_2} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\mu J_2 R_E^2}{2r^3}(3\sin^2 i\ \sin^2 u - 1)\ dM$$

Substituting dM and r:

Since $r = \frac{p}{1+e\cos\nu}$,

$$\frac{1}{r^3} = \left(\frac{1 + e\cos\nu}{p}\right)^3 = \frac{(1 + e\cos\nu)^3}{p^3} \quad (1)$$

Since

$$dM = (1 - e^2)^{3/2} \frac{d\nu}{(1 + e\cos\nu)^2} \quad (2)$$

the product becomes:

$$\frac{1}{r^3}\, dM = \frac{(1 + e\cos\nu)^3}{p^3} \cdot (1 - e^2)^{3/2} \frac{d\nu}{(1 + e\cos\nu)^2} = \frac{(1 - e^2)^{3/2}}{p^3}(1 + e\cos\nu)\, d\nu \quad (1)(2)$$

Thus

$$\overline{\mathcal{R}}_{J_2} = \frac{\mu J_2 R_E^2}{4\pi} \int_0^{2\pi} (3\sin^2 i \sin^2(\omega + \nu) - 1)\, \frac{(1 - e^2)^{3/2}}{p^3}(1 + e\cos\nu)\, d\nu.$$

Factoring out constants w.r.t. ν:

$$\overline{\mathcal{R}}_{J_2} = \frac{\mu J_2 R_E^2}{4\pi} \frac{(1 - e^2)^{3/2}}{p^3} \int_0^{2\pi} (3\sin^2 i \sin^2(\omega + \nu) - 1)\,(1 + e\cos\nu)\, d\nu.$$

Now using $\sin^2(\omega + \nu) = \frac{1}{2}\big(1 - \cos(2\omega + 2\nu)\big)$:

$$3\sin^2 i \sin^2(\omega + \nu) - 1 = 3\sin^2 i \cdot \frac{1}{2}\big(1 - \cos(2\omega + 2\nu)\big) - 1$$

$$= \left(\frac{3}{2}\sin^2 i - 1\right) - \frac{3}{2}\sin^2 i \cos(2\omega + 2\nu).$$

The integral is consequently partitioned into two components, I_1 and I_2:

$$I = \int_0^{2\pi} \left[\left(\frac{3}{2}\sin^2 i - 1\right) - \frac{3}{2}\sin^2 i \cos(2\omega + 2\nu)\right](1 + e\cos\nu)\, d\nu = I_1 + I_2,$$

where

$$I_1 = \left(\frac{3}{2}\sin^2 i - 1\right)\int_0^{2\pi} (1 + e\cos v)\ dv,$$

and

$$I_2 = -\frac{3}{2}\sin^2 i \int_0^{2\pi} \cos\left(2\omega + 2v\right)(1 + e\cos v)\, dv.$$

Computing I_1 exactly:

$$\int_0^{2\pi} (1 + e\cos v)\ dv = \int_0^{2\pi} 1\ dv + e\int_0^{2\pi} \cos v\, dv = 2\pi + e \cdot 0 = 2\pi.$$

So

$$I_1 = \left(\frac{3}{2}\sin^2 i - 1\right) 2\pi.$$

Computing I_2 exactly:

$$I_2 = -\frac{3}{2}\sin^2 i \left[\int_0^{2\pi} \cos\left(2\omega + 2v\right) dv + e\int_0^{2\pi} \cos v \cos(2\omega + 2v)\, dv\right].$$

First sub-integral:

$$\int_0^{2\pi} \cos\left(2\omega + 2v\right) dv = 0 \quad \text{(one full period of a cosine).}$$

Second sub-integral: using product-to-sum:

$$\cos\nu \cdot \cos(2\omega + 2\nu) = \frac{1}{2}[\cos((2\omega + 2\nu) - \nu) + \cos((2\omega + 2\nu) + \nu)]$$

$$= \frac{1}{2}[\cos(2\omega + \nu) + \cos(2\omega + 3\nu)].$$

Now integrating:

$$\int_0^{2\pi} \cos(2\omega + \nu)\, d\nu = 0, \qquad \int_0^{2\pi} \cos(2\omega + 3\nu)\, d\nu = 0.$$

So the entire second sub-integral is 0, hence $I_2 = 0$.

Therefore,

$$I = I_1 + I_2 = 2\pi\left(\frac{3}{2}\sin^2 i - 1\right).$$

Substituting back:

$$\overline{\mathcal{R}}_{J_2} = \frac{\mu J_2 R_E^2}{4\pi}\frac{(1-e^2)^{3/2}}{p^3} \cdot 2\pi\left(\frac{3}{2}\sin^2 i - 1\right) = \frac{\mu J_2 R_E^2}{2}\frac{(1-e^2)^{3/2}}{p^3}\left(\frac{3}{2}\sin^2 i - 1\right).$$

Now substituting $p = a(1 - e^2)$, so:

$$p^3 = a^3(1-e^2)^3, \quad \Rightarrow \quad \frac{(1-e^2)^{3/2}}{p^3} = \frac{(1-e^2)^{3/2}}{a^3(1-e^2)^3} = \frac{1}{a^3(1-e^2)^{3/2}}.$$

Thus:

$$\boxed{\overline{\mathcal{R}}_{J_2} = \frac{\mu J_2 R_E^2}{2a^3(1-e^2)^{3/2}}\left(\frac{3}{2}\sin^2 i - 1\right)}.$$

Equivalent form using $\cos i$. Since $\sin^2 i = 1 - \cos^2 i$:

$$\frac{3}{2}\sin^2 i - 1 = \frac{3}{2}(1 - \cos^2 i) - 1 = \frac{1}{2} - \frac{3}{2}\cos^2 i = \frac{1}{2}(1 - 3\cos^2 i).$$

So:

$$\boxed{\overline{\mathcal{R}}_{J_2} = \frac{\mu J_2 R_E^2}{4a^3(1-e^2)^{3/2}}(1 - 3\cos^2 i)}.$$

Applying Lagrange Planetary Equations and Computing Derivatives Step-by-Step

One common Lagrange planetary equation form for secular node rate (using the averaged disturbing function) is:

$$\dot{\Omega} = -\frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial \overline{\mathcal{R}}}{\partial i},$$

and for perigee:

$$\dot{\omega} = \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial \overline{\mathcal{R}}}{\partial e} - \frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial \overline{\mathcal{R}}}{\partial i}.$$

These are consistent with the standard use of Lagrange planetary equations in orbital perturbation theory; Kozai also writes a set of differential equations for orbital element variations in terms of derivatives of the disturbing function.

Now computing the partial derivatives with chain rule. Let:

$$\overline{\mathcal{R}}_{J_2} = K\,(1 - 3\cos^2 i)\,(1-e^2)^{-3/2}, \quad \text{where} \quad K = \frac{\mu J_2 R_E^2}{4a^3}.$$

Computing $\frac{\partial \overline{\mathcal{R}}}{\partial i}$:

Only $(1 - 3\cos^2 i)$ depends on i.

Differentiating:

Starting with $\cos^2 i$. By chain rule:

$$\frac{d}{di}(\cos^2 i) = 2\cos i \cdot \frac{d}{di}(\cos i) = 2\cos i(-\sin i) = -2\cos i \cdot \sin i$$

Multiplying by -3:

$$\frac{d}{di}(1 - 3\cos^2 i) = -3 \cdot \frac{d}{di}(\cos^2 i) = -3(-2\cos i \sin i) = 6\cos i \cdot \sin i.$$

So:

$$\frac{\partial \overline{\mathcal{R}}}{\partial i} = K\,(1 - e^2)^{-3/2} \cdot 6\cos i \cdot \sin i = 6K\cos i \cdot \sin i(1 - e^2)^{-3/2}.$$

Substituting $K = \mu J_2 R_E^2/(4a^3)$:

$$\frac{\partial \overline{\mathcal{R}}}{\partial i} = 6\left(\frac{\mu J_2 R_E^2}{4a^3}\right)\cos i \cdot \sin i(1 - e^2)^{-3/2} = \frac{3}{2}\frac{\mu J_2 R_E^2}{a^3}\cos i \cdot \sin i(1 - e^2)^{-3/2}.$$

Computing $\frac{\partial \overline{\mathcal{R}}}{\partial e}$:

Only $(1 - e^2)^{-3/2}$ depends on e. Let $f(e) = (1 - e^2)^{-3/2}$.

Chain rule:

Differentiating the outer power:

$$\frac{d}{dx}\left(x^{-3/2}\right) = -\frac{3}{2}x^{-5/2}.$$

Here $x(e) = 1 - e^2$, so $dx/de = -2e$.

Combining:

$$\frac{df}{de} = -\frac{3}{2}(1-e^2)^{-5/2} \cdot (-2e) = 3e(1-e^2)^{-5/2}.$$

Therefore:

$$\frac{\partial \overline{\mathcal{R}}}{\partial e} = K\,(1-3\cos^2 i) \cdot 3e(1-e^2)^{-5/2}.$$

Substituting Into Lagrange Equations and Simplifying to Standard Forms

Deriving $\dot{\Omega}$:

Starting with:

$$\dot{\Omega} = -\frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial \overline{\mathcal{R}}}{\partial i}.$$

Inserting $\frac{\partial \overline{\mathcal{R}}}{\partial i}$:

$$\dot{\Omega} = -\frac{1}{na^2\sqrt{1-e^2}\sin i}\left[\frac{3}{2}\frac{\mu J_2 R_E^2}{a^3}\cos i \cdot \sin i(1-e^2)^{-3/2}\right].$$

Canceling $\sin i$:

$$\dot{\Omega} = -\frac{1}{na^2\sqrt{1-e^2}}\left[\frac{3}{2}\frac{\mu J_2 R_E^2}{a^3}\cos i(1-e^2)^{-3/2}\right].$$

Combining the $(1-e^2)$ factors:

$$\sqrt{1-e^2} \cdot (1-e^2)^{3/2} = (1-e^2)^2,$$

so:

$$\dot{\Omega} = -\frac{3}{2}\frac{\mu J_2 R_E^2}{na^5}\frac{\cos i}{(1-e^2)^2}.$$

Now simplifying using the definition

$$n = \sqrt{\frac{\mu}{a^3}} \quad \Rightarrow \quad \mu = n^2 a^3.$$

So:

$$\frac{\mu}{na^5} = \frac{n^2 a^3}{na^5} = \frac{n}{a^2}.$$

Thus:

$$\dot{\Omega} = -\frac{3}{2}J_2\, n\, \frac{R_E^2}{a^2(1-e^2)^2}\cos i.$$

Since $p = a(1-e^2)$, we have $p^2 = a^2(1-e^2)^2$. Therefore:

$$\boxed{\dot{\Omega} = -\frac{3}{2}J_2\, n\left(\frac{R_E}{p}\right)^2 \cos i}.$$

This matches the commonly used SSO nodal precession formula (NASA also presents it in a simplified $e \approx 0$ form).

Deriving $\dot{\omega}$:

Starting with:

$$\dot{\omega} = \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial\overline{\mathcal{R}}}{\partial e} - \frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial\overline{\mathcal{R}}}{\partial i}.$$

Computing the first term T_1:

$$T_1 = \frac{\sqrt{1-e^2}}{na^2 e}\left[K(1-3\cos^2 i)\cdot 3e(1-e^2)^{-5/2}\right].$$

Canceling e:

$$T_1 = \frac{\sqrt{1-e^2}}{na^2}\cdot 3K(1-3\cos^2 i)(1-e^2)^{-5/2}.$$

Combining $\sqrt{1-e^2}\cdot(1-e^2)^{-5/2} = (1-e^2)^{-2}$:

$$T_1 = \frac{3K}{na^2}(1-3\cos^2 i)(1-e^2)^{-2}.$$

Now substituting $K = \mu J_2 R_E^2/(4a^3)$:

$$T_1 = \frac{3}{na^2}\left(\frac{\mu J_2 R_E^2}{4a^3}\right)(1-3\cos^2 i)(1-e^2)^{-2} = \frac{3\mu J_2 R_E^2}{4na^5}(1-3\cos^2 i)(1-e^2)^{-2}.$$

Using $\mu/(na^5) = n/a^2$:

$$T_1 = \frac{3}{4}J_2 n\frac{R_E^2}{a^2}(1-3\cos^2 i)(1-e^2)^{-2}.$$

Replacing $a^2(1-e^2)^2 = p^2$:

$$T_1 = \frac{3}{4}J_2 n\left(\frac{R_E}{p}\right)^2(1-3\cos^2 i).$$

Computing the second term T_2:

$$T_2 = -\frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial\overline{\mathcal{R}}}{\partial i} = -\frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\left[\frac{3}{2}\frac{\mu J_2 R_E^2}{a^3}\cos i\sin i(1-e^2)^{-3/2}\right].$$

Canceling $\sin i$ and combine $\sqrt{1-e^2}(1-e^2)^{3/2} = (1-e^2)^2$:

$$T_2 = -\frac{3}{2}\frac{\mu J_2 R_E^2}{na^5}\frac{\cos^2 i}{(1-e^2)^2} = -\frac{3}{2}J_2 n\frac{R_E^2}{a^2}\frac{\cos^2 i}{(1-e^2)^2} = -\frac{3}{2}J_2 n\left(\frac{R_E}{p}\right)^2 \cos^2 i.$$

Now adding:

$$\dot{\omega} = T_1 + T_2 = \frac{3}{4}J_2 n\left(\frac{R_E}{p}\right)^2 (1-3\cos^2 i) - \frac{3}{2}J_2 n\left(\frac{R_E}{p}\right)^2 \cos^2 i.$$

Factoring $\frac{3}{4}J_2 n\left(\frac{R_E}{p}\right)^2$:

$$\dot{\omega} = \frac{3}{4}J_2 n\left(\frac{R_E}{p}\right)^2 [(1-3\cos^2 i) - 2\cos^2 i] = \frac{3}{4}J_2 n\left(\frac{R_E}{p}\right)^2 [1-5\cos^2 i].$$

So:

$$\boxed{\dot{\omega} = \frac{3}{4}J_2 n\left(\frac{R_E}{p}\right)^2 (1-5\cos^2 i)}.$$

Many textbooks present $\dot{\omega}$ as

$$\dot{\omega} = \frac{3}{4}J_2 n\left(\frac{R_E}{p}\right)^2 (5\cos^2 i - 1),$$

which is the same expression with the sign reversed by re-ordering the bracket. The physical sign depends on inclination: for near-polar SSOs ($i \approx 98°$), $\cos^2 i$ is small so $(5\cos^2 i - 1) < 0$, hence $\dot{\omega}$ is negative, matching my numerical results below.

Unit Conversion to deg/day

Rates computed in rad/s are converted to deg/day using the following conversion factor:

$$\left(\frac{\text{rad}}{\text{s}}\right)\cdot\left(\frac{180°}{\pi\ \text{rad}}\right)\cdot\left(\frac{86400\ \text{s}}{1\ \text{day}}\right) = \left(\frac{86400\cdot 180}{\pi}\right)\frac{\text{deg}}{\text{day}}.$$

Numerical Evaluation and Sensitivity Analysis

TLE Parameters and Conversion Methodology for Model Inputs

From a TLE:

- Parameters such as i, Ω, e, and ω are read directly from line 2 of the TLE. Mean motion is then converted to SI units, from which a is derived using

$$n_{\text{rad/s}} = n_{\text{rev/day}} \cdot \frac{2\pi}{86400}.$$

- a is computed from $n = \sqrt{\mu/a^3} \Rightarrow a = (\mu/n^2)^{1/3}$.

- $p = a(1 - e^2)$ is computed.

TLE field meanings (including that the "first derivative of mean motion" is stored as "mean motion derivative divided by two") are documented by CelesTrak.

Mission Case Inputs From TLEs

Earth constants used (WGS84 set in NAIF kernel): R_E, J_2, J_3.

TLE sources:

- Landsat 8 TLE (epoch 2026-03-04): CelesTrak.

- Sentinel-2A TLE (epoch 2026-02-25): CelesTrak.

These are consistent with each mission's known SSO altitude range.

Numerical Results for Landsat 8 and Sentinel-2A

TLE-Derived Orbital Parameters and Derived a, p

Satellite	TLE epoch (UTC)	i (deg)	Ω (deg)	e	ω (deg)	n (rev/day)	a (km)	$a - R_E$ (km)	p (km)
Landsat 8	2026-03-04 18:50:15	98.1927	135.5908	0.0001238	91.0172	14.5712596	7080.606	702.469	7080.606
Sentinel-2A	2026-02-25 04:19:12	98.5024	132.8928	0.0054368	194.7437	14.3019292	7169.223	791.086	7169.011

Mission-orbit context: Landsat 8 is documented at ~705 km altitude and $98.2°$ inclination. Sentinel-2 is documented at 786 km sun-synchronous altitude and ~100 min period.

Computed Secular Rates Due to J_2

Satellite	$\dot{\Omega}$ (rad/s)	$\dot{\Omega}$ (deg/day)	$\dot{\omega}$ (rad/s)	$\dot{\omega}$ (deg/day)
Landsat 8	1.98978×10^{-7}	$+0.985011$	-6.27267×10^{-7}	-3.10519
Sentinel-2A	1.97663×10^{-7}	$+0.978501$	-5.95392×10^{-7}	-2.94740

Physically, for retrograde SSOs ($i > 90°$), $\cos i < 0$, so the node rate $\dot{\Omega}$ becomes positive: Ω increases over time. This is exactly what enables sun-synchronicity when tuned near the Sun's motion.

- The perigee rate $\dot{\omega}$ can be positive or negative depending on i. Near-polar orbits often yield $(5\cos^2 i - 1) < 0$, giving negative $\dot{\omega}$.

Sensitivity Analysis with Tables and Plots

Individual parameters are varied while maintaining the TLE baseline for all other variables:

- a: $a_0 \pm 50$ km

- e: from 0 to 0.01

- i: $i_0 \pm 1°$

Landsat 8 Sensitivity Tables

Vary a:

Δa (km)	a (km)	$\dot{\Omega}$ (deg/day)	$\dot{\omega}$ (deg/day)
-50	7030.606	1.009748	-3.183176
-25	7055.606	0.997281	-3.143874
0	7080.606	0.985011	-3.105194
+25	7105.606	0.972935	-3.067124
+50	7130.606	0.961048	-3.029652

Vary e: (effect is small here because e^2 appears in $p = a(1 - e^2)$)

e	p (km)	$\dot{\Omega}$ (deg/day)	$\dot{\omega}$ (deg/day)
0.000	7080.606	0.985011	-3.105194
0.002	7080.578	0.985019	-3.105219
0.004	7080.493	0.985043	-3.105293
0.006	7080.351	0.985082	-3.105418
0.008	7080.153	0.985137	-3.105592
0.010	7079.898	0.985208	-3.105815

Vary i: (dominant because of $\cos i$ and $\cos^2 i$)

i (deg)	$\cos i$	$\dot{\Omega}$ (deg/day)	$\dot{\omega}$ (deg/day)
97.1927	-0.125207	0.865457	-3.185208
98.1927	-0.142503	0.985011	-3.105194
99.1927	-0.159755	1.104265	-3.015081

Sentinel-2A sensitivity tables

Vary a:

Δa (km)	a (km)	$\dot{\Omega}$ (deg/day)	$\dot{\omega}$ (deg/day)
-50	7119.223	1.002766	-3.020491
-25	7144.223	0.990538	-2.983659
0	7169.223	0.978501	-2.947402
+25	7194.223	0.966652	-2.911710
+50	7219.223	0.954986	-2.876571

Vary e:

e	p (km)	$\dot{\Omega}$ (deg/day)	$\dot{\omega}$ (deg/day)
0.000	7169.223	0.978443	-2.947228
0.002	7169.194	0.978451	-2.947251
0.004	7169.108	0.978475	-2.947325
0.006	7168.965	0.978515	-2.947450
0.008	7168.764	0.978570	-2.947624
0.010	7168.506	0.978641	-2.947848

Vary i:

i (deg)	$\cos i$	$\dot{\Omega}$ (deg/day)	$\dot{\omega}$ (deg/day)
97.5024	-0.130480	0.895037	-3.039085
98.5024	-0.147694	0.978501	-2.947402
99.5024	-0.164863	1.061667	-2.845490

Model Validation and Uncertainty Analysis

Algebraic Validation Against Published Forms

Node precession: NASA mission-operations material states the dominant even-zonal term yields (for small e) a node rate

$$\frac{d\Omega}{dt} = -\frac{3}{2}\sqrt{\mu}\, R_E^2 J_2 \frac{\cos i}{a^{7/2}},$$

which is exactly my $\dot{\Omega}$ formula when I set $p \approx a$ (for instance, $e \approx 0$) and expand $n = \sqrt{\mu/a^3}$ to show $n/a^2 = \sqrt{\mu}/a^{7/2}$.

Method consistency: Kozai's classic paper includes the disturbing function for an axially symmetric Earth field and writes the equations of variation of the orbital elements in terms of the derivatives of this disturbing function, which is the same method as I follow

(disturbing function → variational equations → secular drift).

Numerical Cross-Check Using TLE-to-TLE Drift

Empirical $\dot{\Omega}$ for Landsat 8 From Two Epochs

The empirical node drift from TLEs is estimated using the following relationship:

$$\dot{\Omega}_{\text{emp}} \approx \frac{\Omega_2 - \Omega_1}{t_2 - t_1},$$

with wrap-around handling for angles near $0°/360°$.

Two publicly accessible TLE epochs are utilized for this cross-check:

- Epoch 1 (CelesTrak): 2026-03-04 18:50:15 UTC, $\Omega_1 = 135.5908°$.

- Epoch 2 (SatelliteMap): 2026-03-05 04:43:33 UTC, $\Omega_2 = 135.9961°$.

Time difference:

$$\Delta t \approx 0.4120 \text{ days}, \quad \Delta\Omega = 0.4053°,$$

so:

$$\dot{\Omega}_{\text{emp}} \approx \frac{0.4053}{0.4120} \approx 0.984 \text{ °/day}.$$

The J_2-only model prediction for Landsat 8 at epoch 1 is $0.9850°$/day, agrees with empirical data to within about $0.0013°$/day over this short arc (about 0.13%). This supports the correctness of the $\dot{\Omega}$ derivation and the numerical pipeline.

Why $\dot{\omega}$ is Hard to Validate for Landsat 8 from TLEs

Because of the very small eccentricity ($e \sim 10^{-4}$) of Landsat 8, the location of perigee is poorly defined. For a near-circular orbit, a rotation of the line of apsides results in a negligible change to the spacecraft's state vector, making the argument of perigee physically ill-defined. Furthermore, Kozai's lemma indicates that if the eccentricity of an orbit is very small, certain terms that relate to the perigee of that orbit become very large. Thus, while $\dot{\omega}$ is mathematically correct from the derivation of the TLE, it is not a stable benchmark against which to measure drift in the TLE for a mission like Landsat 8, which orbits the Earth with such a small eccentricity.

Sentinel-2A Empirical Drift Method and Data Availability

For Sentinel-2A, ω is better-conditioned because $e \approx 0.0054$. However, robust drift estimation requires two epochs sufficiently separated in time. While there are repositories of TLE data that are publicly available (like CelesTrak), they only provide the current/latest set of elements unless one utilizes an archive service (for example, Space-Track GP_History). Consequently, while the methodology is detailed in this essay, the empirical values for Sentinel-2A are omitted due to data constraints.

Quantifying Neglected Effects and Providing Error Bars

Magnitude of J_3 Relative to J_2

Earth's J_3 coefficient is about 2.34×10^{-3} of J_2 in coefficient magnitude:

$$\frac{|J_3|}{J_2} \approx 0.00234.$$

Using a rough scaling for the potential contribution at LEO where $R_E/r \approx 0.9$,

$$\left|\frac{U_{J_3}}{U_{J_2}}\right| \sim \frac{|J_3|}{J_2}\frac{R_E}{r} \approx 0.0021 \quad \text{(about } 0.21\%\text{)}.$$

This supports truncating at J_2 for a first-order precession model, but also motivates including an uncertainty band of order 0.2%–0.3% on rates attributed purely to geopotential truncation.

Bounding Drag from the TLE Mean-Motion Derivative

CelesTrak documents that TLE field 1.9 is "mean motion derivative divided by two." Let n be mean motion. Since $n \propto a^{-3/2}$, differentiating $\ln n = -3/2 \ln a$ gives:

$$\frac{\dot{n}}{n} = -\frac{3}{2}\frac{\dot{a}}{a} \quad \Rightarrow \quad \dot{a} = -\frac{2}{3}a\frac{\dot{n}}{n}.$$

Utilizing the mean motion derivative $\dot{n}$ from the TLE, an order-of-magnitude decay rate is obtained:

- Landsat 8: $\dot{a} \approx -3.7$ m/day (very small).
- Sentinel-2A: $\dot{a} \approx -1.1$ m/day.

Since $\dot{\Omega} \sim a^{-7/2}$ for small e, a day of drag-induced change in a changes $\dot{\Omega}$ by only $O(10^{-6})$ fraction per day for these well-maintained missions, in other words, negligible for short validation windows.

Final Uncertainty Intervals and Model Accuracy

An uncertainty interval, dominated by geopotential truncation, is assigned:

- Geopotential truncation (omit J_3 and higher): $\pm 0.25\%$ on rates (conservative vs my ~0.21% estimate).
- Drag-induced change in a near epoch: $< \pm 0.01\%$ over short multi-day windows for these missions.

The resulting model-only values are reported as:

- Landsat 8: $\dot{\Omega} = 0.9850 \pm 0.0025\ ^\circ/$day, $\dot{\omega} = -3.105 \pm 0.008\ ^\circ/$day.
- Sentinel-2A: $\dot{\Omega} = 0.9785 \pm 0.0024\ ^\circ/$day, $\dot{\omega} = -2.947 \pm 0.007\ ^\circ/$day.

These are not measurement uncertainties, but model-structure uncertainties from neglected physics in a J_2-only theory.

Conclusion

Reflecting on this investigation, the most challenging aspect transitioning from the ideal model of Kepler to the perturbing potential of an oblate Earth. Deriving the secular rates for $\boldsymbol{\Omega}$ and $\boldsymbol{\omega}$ required a multi-step process involving calculus and integrals. The algebra involved in the Lagrange planetary equations was difficult for me, but breaking down the derivation into individual steps allowed me to better understand the derivation of those equations.

For the empirical data component of the investigation, I found it to be more straightforward than the derivation of the secular rates. The availability of satellite data allowed me to easily perform my observations. One major improvement I made to the essay was including a sensitivity analysis regarding the parameters of Landsat 8 and Sentinel-2A. I initially planned to calculate the secular rates for each satellite, but performing a sensitivity analysis of their orbital parameters allowed me to gain a more thorough understanding of their mission designs.

One of the challenges I encountered in the validation step was the behavior of Landsat 8's argument of perigee. I spent considerable effort attempting to validate the value of $\dot{\Omega}$ for Landsat 8, but I found that due to the satellite's small eccentricity, its perigee was ill-defined. I had to abandon this effort and instead perform an uncertainty analysis regarding the model's structure.

Overall, though challenging, this investigation allowed me to learn a lot about the role of Earth's oblateness in satellite orbits, and how mathematical abstraction can be utilized to understand physical phenomena.

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Appendix with Exact TLE Lines Used

CelesTrak TLEs used for computation:

LANDSAT 8 (CelesTrak, 2LE)

1 39084U 13008A 26063.78490213 .00000573 00000+0 13707-3 0 9992

2 39084 98.1927 135.5908 0001238 91.0172 269.1168 14.57125961682692

SENTINEL-2A (CelesTrak, 2LE)

1 40697U 15028A 26056.18000490 .00000159 00000+0 76852-4 0 9999

2 40697 98.5024 132.8928 0054368 194.7437 165.2162 14.30192923557712

Second Landsat 8 epoch used for empirical $\dot{\Omega}$ estimate:

LANDSAT 8 (SatelliteMap.space)

1 39084U 13008A 26064.19691556 .00000495 00000-0 11983-3 0 9999

2 39084 98.1926 135.9961 0001236 91.3426 268.7914 14.57126036682752

Appendix B: Reproducibility Notes

- For all computations I used the standard conversions described in the "TLE format" documentation (mean motion in rev/day, mean-motion derivative stored as half).
- For orbits with extremely small eccentricity, ω-based empirical drift is treated with caution due to the known small-e singular behavior discussed in classical perturbation theory.