INTERNATIONAL BACCALAUREATE DIPLOMA PROGRAM

MATHEMATICS EXTENDED ESSAY

DEVELOPING A MATHEMATICAL MODEL OF RADIOACTIVE DECAY CHAINS WITH BRANCHING PROBABILITIES USING MATRIX ALGEBRA AND LAPLACE TRANSFORM

RESEARCH QUESTION: *How can matrix algebra and Laplace transform be utilized for developing an improved mathematical model for radioactive decay chains that have branching probabilities?*

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I. INTRODUCTION

Radioactive decay refers to the spontaneous transition of an unstable nucleus to a more stable nucleus [1]. Therefore, it explains the distribution of the different types of atoms over a certain amount of time. Additionally, it can also refer to energy release during the transition process of the nucleus.

Consequently, these applications are crucial for the safety of living organisms, and they provide the foundation for nuclear physics. As a result of the significance that radioactive decay held in physics, several investigations were conducted to enhance the comprehension of the processes of decay, and these investigations led to the discovery that radioactive decay can occur through several distinct successive transformations[2]; thus, the mathematical models for describing the change in terms of abundances of atoms over a certain time became complicated.

Commonly utilized mathematical model for describing the behavior of radioactive decay chains, in terms of abundances of different atoms in a dynamical system of time, is known as Bateman Equation which is the combination of several first-order differential equations representing the individual co-occurring decays. However, this model has significant

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[1] D. Ilem-Ozdemir, E. A. Gundogdu, M. Ekinci, E. Ozgenc, and M. Asikoglu, “Nuclear medicine and radiopharmaceuticals for molecular diagnosis,” in *Biomedical Applications of Nanoparticles*, A. M. Grumezescu, Ed. Norwich, NY: William Andrew Publishing, 2019, ch. 17, pp. 457–490. doi: 10.1016/B978-0-12-816506-5.00017-6.

[2] E. Rutherford, "Theory of successive changes," in *Radio-activity*, 2nd ed. Cambridge, U.K.: Cambridge Univ. Press, 1905, ch. 9.

limitations and flaws such as the absence of decaying branches and failing to maintain accuracy for certain cases [3].

Upon the recognition of these limitations and flaws of the Bateman Equation, a new and enhanced mathematical model was determined to be essential. This investigation aims to develop an improved mathematical model for describing radioactive decay chains, in terms of generality and computational efficiency. To accomplish the desired model, mathematical concepts of calculus and linear algebra will be utilized. The foundation for this model is Rutherford's equation, which describes the change in the abundance of an atom type over a period in a singular decay process with a first-order differential equation [4]. This equation will be modified to allow for the representation of possible decay branches. Then, these equations will be combined to represent the full extent of the chain with the use of matrices, as they can efficiently represent coupled first-order differential equations. Finally, Laplace transform, a common method for obtaining solutions of first-order differential equations, will be used to obtain the final expression for the model to allow for reduced computational complexity.

II. Research Question

"How can matrix algebra and Laplace transform be utilized for developing an improved mathematical model for radioactive decay chains that have branching probabilities?"

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[3] L. J. Harr, "Precise calculation of complex radioactive decay chains," M.S. thesis, Dept. Eng. Phys., Air Force Inst. of Technol., Wright-Patterson AFB, OH, USA, 2007. [Online]. Available: https://scholar.afit.edu/etd/2924. Accessed: Sept. 10, 2025.

[4] E. Rutherford, "Radioactive changes in thorium," in *Radioactive Transformations*, London, U.K.: Constable, 1906, ch. 3, pp. 43–44.

III. Foundations of Radioactive Decay Models

A. Radioactive Decay

In quantum mechanics, radioactive decay is recognized as a randomly occurring spontaneous process of an unstable atom transitioning into a more stable atom by emitting energy from its nucleus [5][6]. Generally, during the decay, nucleus of the atom releases subatomic particles. As a result, the disintegrating nucleus (parent nucleus) will transform into a different type of nucleus (daughter nucleus). An example of this process is shown in figure 1, which shows uranium-238, a common isotope of the substance uranium, decaying into thorium-234, an isotope of the substance thorium, by emitting a combination of subatomic particles known as α particles.

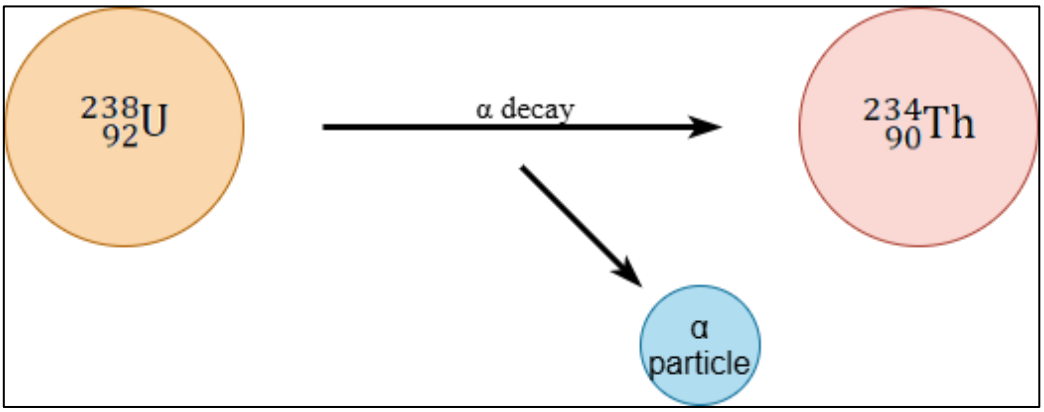

Fig. 1. Diagram of uranium-238 decaying into thorium-234 by releasing α particle [7]

[5] E. Rutherford, "Radio-active processes," in *Radio-activity*, 2nd ed. Cambridge, U.K.: Cambridge Univ. Press, 1905, ch. 8.

[6] M. G. Stabin, Radiation Protection and Dosimetry: An Introduction to Health Physics. New York, NY, USA: Springer, 2007. doi: 10.1007/978-0-387-49983-3.

[7] "Diagram- draw.io." draw.io. https://app.diagrams.net (accessed November 13, 2025).

In this paper, we use this information while interpreting the radioactive decay chains so that we can construct a mathematical foundation for it.

B. Rutherford's Formulation for Radioactive Decay

Ernest Rutherford revealed that with a significant number of atoms in the system, radioactive decay follows a predictable and statistical pattern [4] [8]. Upon his examinations, he proposed the following mathematical equation for describing the decay rate, $\Lambda(t)$, of a substance in a homogeneous system where no chemical reaction occurs:

$$\Lambda(t) = \frac{dN(t)}{dt} = -\lambda N(t)$$

where $N(t)$ is the number of atoms the substance present at time $t \in \mathbb{R}^+ \cup \{0\}$, and $\lambda \in \mathbb{R}^+ \cup \{0\}$ is a distinct decay constant of the substance [9], meaning that each radioactive substance has its own $\lambda > 0$ that is not equal to any other radioactive substance's decay constant, while all stable (non-radioactive) substance share the same value $\lambda = 0$. In this paper, this equation for decay process serves as the fundamental basis for the mathematical model of radioactive decay chain that we develop.

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[4] E. Rutherford, "Radioactive changes in thorium," in *Radioactive Transformations*, London, U.K.: Constable, 1906, ch. 3, pp. 43–44.

[8] E. Rutherford, "Radio-active Substances," in *Radio-activity*, 2nd ed. Cambridge, U.K.: Cambridge Univ. Press, 1905, ch. 1.

[9] E. Rutherford, "Radioactive changes in thorium," in *Radioactive Transformations*, London, U.K.: Constable, 1906, ch. 3, p. 45.

C. Theory of Successive Changes

Theory of Successive Changes refers to the phenomenon where the process radioactive decay occurs through several successive changes [2], and it is known as radioactive decay chain. It occurs when an unstable substance decays into another unstable substance, thus, forming a chain of decays. These systems can be mathematically represented as a combination of multiple consecutive modified decay formulas:

$$\frac{dN_1(t)}{dt} = -\lambda_1 N_1(t)$$

$$\frac{dN_2(t)}{dt} = \lambda_1 N_1(t) - \lambda_2 N_2(t)$$

$$\vdots$$

$$\frac{dN_n(t)}{dt} = \lambda_{n-1} N_{n-1}(t) - \lambda_n N_n(t)$$

where λ_i is the decay constant for the i^{th} substance, $n \in \mathbb{N}^+$ is the number of different substances that are in the system, and $N_i(t)$ is the number of atoms of the i^{th} substance present at time t which is, for all t, $N_i(t) \geq 0$. The standard solution for the abundance of a substance n is known as Bateman Equation [10]:

[2] E. Rutherford, "Theory of successive changes," in *Radio-activity*, 2nd ed. Cambridge, U.K.: Cambridge Univ. Press, 1905, ch. 9.

[10] H. Bateman, "The solution of a system of differential equations occurring in the theory of radioactive transformations," Proc. Cambridge Philos. Soc., vol. 15, no. 5, pp. 423–427, 1910.

$$N_n(t) = N_1(0) \times \left(\prod_{i=1}^{n-1} \lambda_i\right) \times \sum_{i=1}^{n} \frac{e^{-\lambda_i t}}{\prod_{j=1, j\neq i}^{n}(\lambda_j - \lambda_i)}$$

However, this formula doesn't incorporate a variable for branching probabilities that is explained at Section III-D. Furthermore, it assumes that the initial abundances of substances other than N_1 are 0, meaning that it is not usable in such conditions where other substances have abundances larger than 0. In this paper, we aim to develop a more general model that can handle the specific conditions where classical Bateman Equation is not applicable.

D. Branching Probabilities in Radioactive Decay Chains

If the i^{th} radioactive substance in the system, which has n distinct isotopes, has probability decaying into more than one substance, then we have set of probabilities which can be represented as

$$p_{i\to j} \in [0,1], for\ all\ i, \sum_{j=1}^{n} p_{i\to j} \leq 1$$

where j represents the j^{th} substance that i^{th} substance can disintegrate into, and $p_{i\to j}$ represents the probability of i^{th} substance decaying into the j^{th} substance. This can significantly impact the abundances of the substances found in that system. Thus, in order to develop a precise

mathematical model, we should not assume that branching probabilities are absent in the system.

IV. Mathematical Key

A. Laplace Transform

The Laplace Transform is an integral that converts a function of time into a function of a complex variable s [11], and it is denoted with "$\mathcal{L}$". The Laplace transform is defined as

$$\mathcal{L}(f;s) = \int_0^\infty e^{-st} f(t) dt$$

where $f(t)$ is a function of time and the integral converges for the determined values of s [11]. The viability of Laplace transform comes from its ability to convert ordinary differential equations into algebraic equations in the complex plane and its invertibility. Additionally, it behaves linearly during the sum and difference of two functions and multiplication of constants [12]. These properties make Laplace transform a powerful tool for analyzing coupled first-

[11] J. Orloff, "Laplace transform (Topic 12 notes)," 18.04 Complex Variables with Applications, Massachusetts Institute of Technology, Cambridge, MA, USA, Spring 2018. [Online]. Available: https://ocw.mit.edu/courses/18-04-complex-variables-with-applications-spring-2018/1b1cbc6540b338b58b7bc1b1cc1f280b_MIT18_04S18_topic12.pdf. Accessed: Mar. 15, 2025.

[12] S. H. Zak, "The Laplace transform review," ECE 382 course notes, School of Electrical and Computer Engineering, Purdue Univ., West Lafayette, IN, USA, Fall 2012. [Online]. Available: https://engineering.purdue.edu/~zak/hand_2.pdf. Accessed: Dec. 14, 2025.

order differential equations [13]; thus, in this paper, we utilize it to solve radioactive decay chains.

B. Inverse Laplace Transform

Inverse Laplace transform is an operator that converts a function of a complex variable s into a function of desired variables, often time t. The standard notation of inverse Laplace transform is "$\mathcal{L}^{-1}$", and it is defined as

$$\mathcal{L}^{-1}\{\mathcal{L}(f;s)\}(t) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT} e^{st}\mathcal{L}(f;s)ds$$

where $\gamma \in \mathbb{R}$ and $\Re(\gamma) > \Re(a_k)$ for all k, with a_k representing the k^{th} pole of $\mathcal{L}(f;s)$. This formula is also known as Mellin's inversion formula or Bromwich Integral, and to compute this integral, we need to use complex analysis methods such as Cauchy's residue theorem. In this paper, we use Mellin's inversion formula to obtain solutions for the expression found after the Laplace transform.

[13] "Section 12: Laplace transforms," Mathematical Methods lecture notes, Univ. Edinburgh, Edinburgh, U.K., 2005. [Online]. Available: https://www2.ph.ed.ac.uk/~mevans/amm/section12.pdf. Accessed: Nov. 16, 2025.

C. Complex Analysis Tools

1) Path Integrals: Integrating a function $f(x)$, $x \in \mathbb{R}$, along the path of a real line from a to b gives $\int_a^b f(x)dx$ which has an implicit path that does not require us to specify. However, integrating a function $f(s)$, $s = \alpha + i\omega$ where $\alpha, \omega \in \mathbb{R}$, in a complex plane from complex point a to b gives multiple possible paths. Thus, we specify the contour or the path [14]. The standard notation used for describing this integral is

$$\int_C f(s)ds$$

where C refers to the specific contour which we integrate $f(s)$ along. If C is a closed contour, then we can use the

$$\oint_C f(s)ds$$

which specifically indicates that $f(s)$ is being integrated along a closed contour [14]. In this paper, we use these notations while evaluating the Bromwich integral through Cauchy's residue theorem.

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[14] R. E. Hunt, "Contour integration and transform theory," ch. 5, NST Mathematical Methods II lecture notes, Univ. Cambridge, Cambridge, U.K., 2002. [Online]. Available: https://www.damtp.cam.ac.uk/user/reh10/lectures/nst-mmii-chapter5.pdf. Accessed: Dec. 29, 2025.

2) Singularities and Poles: Consider a function $f(x) = \frac{1}{h(x)}$, $x \in \mathbb{R}$. When $h(\mathrm{x}) = 0$, $f(\mathrm{x})$ is undefined. Similarly, in complex analysis for a function $f(s) = \frac{1}{h(s)}$, $s = \alpha + i\omega$ and $\alpha, \omega \in \mathbb{R}$, when $h(s) = 0$, $f(s)$ is undefined at that complex point which is referred to as a singularity.

Let $\tau \in \mathbb{R}$ such that $h(\tau) = 0$. If $\lim_{x \to \tau^+} f(x) = \pm\infty$ or $\lim_{x \to \tau^-} f(x) = \pm\infty$, then $x = \tau$ is a vertical asymptote of the function $f(x)$.

Let $\varphi \in \mathbb{C}$ such that $h(\varphi) = 0$. If $\lim_{s \to \varphi} |f(s)| = \infty$ then $s = \varphi$ is a pole of the function $f(s)$ [15]. We know that near the complex point φ, the pole of $f(s)$, the function $f(s)$ behaves proportionally to $\frac{1}{(s-\varphi)^m}$, where $m \in \mathbb{N}^+$ and refers to the order of the pole. These concepts are essential to understand the inverse Laplace transform and Cauchy's residue theorem which are the main methods that we employ to solve our Laplace transform.

3) Residue: In the previous section, we said that for a function $f(s)$, $s = \alpha + i\omega$ and $\alpha, \omega \in \mathbb{R}$, which has a pole at the complex point φ, it behaves proportionally to $\frac{1}{(s-\varphi)^m}$, where $m \in \mathbb{N}^+$ and refers to the order of the pole. At the complex point φ, residue of $f(s)$, refers to the coefficient of $\frac{1}{(s-\varphi)}$ which is expressed as $\mathrm{Res}(f, \varphi)$. The standard method employed for

[15] J. Orloff, "Residue theorem (Topic 8 notes)," 18.04 Complex Variables with Applications, Massachusetts Institute of Technology, Cambridge, MA, USA, Spring 2018. [Online]. Available: https://ocw.mit.edu/courses/18-04-complex-variables-with-applications-spring-2018/3c58ce71ec6f7573804221d76158a0bd_MIT18_04S18_topic8.pdf. Accessed: Nov. 11, 2025.

finding the $\mathrm{Res}(f, \varphi)$ is applying Laurent expansion to $f(s)$. However, in this paper, we utilize the standard limit formula in order to compute the residue at a pole of order m which is [16],

$$\mathrm{Res}(f, \varphi) = \frac{1}{(m-1)!} \lim_{s \to \varphi} \frac{d^{m-1}}{ds^{m-1}} \left((s-\varphi)^m f(s) \right)$$

In this paper, we use this limit formula in order to obtain the final solution for our Laplace transform.

4) Cauchy's Residue Theorem: Cauchy's residue theorem states that if a function $f(s)$, $s \in \mathbb{C}$, is holomorphic (analytic) on and inside a closed contour C [17] except its finite number of poles inside C, then

$$\oint_C f(s)\, ds = 2\pi i \sum \mathrm{Res}(f, a_k)$$

where a_k refers to the k^{th} pole of f inside C. As previously mentioned in Section IV-B, we use this theorem to determine an expression for inverse Laplace transform.

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[16] N. Srivastava, "Laurent series and residue calculus," lecture notes, Dept. Math., Univ. California, Berkeley, CA, USA, Mar. 19, 2015. [Online]. Available: https://math.berkeley.edu/~nikhil/courses/121a/laurent.pdf. Accessed: Dec. 30, 2025.

[17] J. Orloff, "Definite integrals using the residue theorem (Topic 9 notes)," 18.04 Complex Variables with Applications, Massachusetts Institute of Technology, Cambridge, MA, USA, Spring 2018. [Online]. Available: https://ocw.mit.edu/courses/18-04-complex-variables-with-applications-spring-2018/f69bc227ae476839819fb18e5b283072_MIT18_04S18_topic9.pdf. Accessed: Dec. 21, 2025.

V. Derivation of Mathematical Model of Radioactive Decay Chains With Branching Probabilities

The mathematical model for describing the abundances of different substances in a radioactive decay chain that have branching probabilities is aimed to be derived in this section. To achieve the desired model in a concise and systematic manner, the derivation is conducted through several stages, which include:

1. Determining the formulation for radioactive decay with branching probabilities.

2. Expressing the determined formulation in matrix notation; hence, the algebraic and computational capabilities of the model are enhanced.

3. Applying Laplace Transform to provide an analytical solution for the system.

A. Formulation for Radioactive Decay with Branching Probabilities

As mentioned in Section III-D, branching is an observable phenomenon in radioactive decay chains. However, the standard formulation for radioactive decay chains does not incorporate a variable for branching probabilities. Therefore, in this section, we modify Rutherford's law of decay to include a variable for branching probabilities.

Let n be the number of distinct isotopes that exist in a homogeneous system, where no chemical reaction occurs. According to Rutherford's law of decay, decay rate of the i^{th} isotope is

$$\Lambda_i(t) = \lambda_i N_i(t) \tag{1}$$

where $\Lambda_i(t)$ refers to the decay rate of i^{th} isotope at time t.

If the i^{th} isotope decays into j^{th} isotope in the system with a probability of $p_{i\to j}$, the expected contribution of i^{th} isotope's decay rate to the j^{th} isotope's growth rate at time t is

$$p_{i\to j}\Lambda_i(t) \tag{2}$$

Thus, we can determine the expected rate of growth of the j^{th} isotope by summing up the expected contribution of every isotope in the system to the j^{th} isotope's growth rate:

$$p_{1\to j}\Lambda_1(t) + p_{2\to j}\Lambda_2(t) + p_{3\to j}\Lambda_3(t) + \cdots + p_{n\to j}\Lambda_n(t) \tag{3}$$

Furthermore, we can express this summation with sigma notation:

$$\sum_{i=1}^{n} p_{i\to j}\Lambda_i(t) \tag{4}$$

Therefore, the j^{th} isotope's rate of change at time t is the difference between its growth and decay rate:

$$\left(\sum_{i=1}^{n} p_{i\to j}\Lambda_i(t)\right) - \Lambda_j(t) \tag{5}$$

Additionally, we can substitute $\Lambda(t)$ with λN to obtain the final formula for radioactive decay with branching probabilities:

$$\frac{dN_j(t)}{dt} = \left(\sum_{i=1}^{n} p_{i\to j}\lambda_i N_i(t)\right) - \lambda_j N_j(t) \tag{6}$$

Now, we have derived the formula for radioactive decay with branching probabilities. This formula explains the relationship between every isotope found in the system and the j^{th} isotope in the system in terms of rate of change. Thus, in this section, we have established the foundation for our model.

B. Construction of Generalized Model of Radioactive Decay Chains with Matrix Notation

In this section, we construct the system by combining the Theory of Successive Changes with the modified formula (6) derived in the previous section. This is done to express

the physical system as a mathematical model. Next, we use matrix notation and algebra to obtain a more compact expression for our model.

Let n be the number of distinct substances that exist in a homogeneous system, where no chemical reaction occurs. Then, this system can be represented by combining the formulation (6), which was derived in the first section:

$$\begin{cases} \frac{dN_1(t)}{dt} = \left(\sum_{i=1}^{n} p_{i\to 1}\lambda_i N_i(t)\right) - \lambda_1 N_1(t) \\ \frac{dN_2(t)}{dt} = \left(\sum_{i=1}^{n} p_{i\to 2}\lambda_i N_i(t)\right) - \lambda_2 N_2(t) \\ \frac{dN_3(t)}{dt} = \left(\sum_{i=1}^{n} p_{i\to 3}\lambda_i N_i(t)\right) - \lambda_3 N_3(t) \\ \vdots \\ \frac{dN_n(t)}{dt} = \left(\sum_{i=1}^{n} p_{i\to n}\lambda_i N_i(t)\right) - \lambda_n N_n(t) \end{cases} \tag{7}$$

Considering that the system consists of n similarly structured, interdependent, equations, it is reasonable to utilize matrix notation. Thus, we can express the system as:

$$\begin{bmatrix} \frac{dN_1(t)}{dt} \\ \frac{dN_2(t)}{dt} \\ \frac{dN_3(t)}{dt} \\ \vdots \\ \frac{dN_n(t)}{dt} \end{bmatrix} = \begin{bmatrix} \left(\sum_{i=1}^{n} p_{i\to 1}\lambda_i N_i(t)\right) - \lambda_1 N_1(t) \\ \left(\sum_{i=1}^{n} p_{i\to 2}\lambda_i N_i(t)\right) - \lambda_2 N_2(t) \\ \left(\sum_{i=1}^{n} p_{i\to 3}\lambda_i N_i(t)\right) - \lambda_3 N_3(t) \\ \vdots \\ \left(\sum_{i=1}^{n} p_{i\to n}\lambda_i N_i(t)\right) - \lambda_n N_n(t) \end{bmatrix} \tag{8}$$

In order to obtain a more compact notation, we can rewrite the left-hand side of the equation using the linear property of differentiation:

$$\begin{bmatrix} \frac{dN_1(t)}{dt} \\ \frac{dN_2(t)}{dt} \\ \frac{dN_3(t)}{dt} \\ \vdots \\ \frac{dN_n(t)}{dt} \end{bmatrix} = \frac{d}{dt} \begin{bmatrix} N_1(t) \\ N_2(t) \\ N_3(t) \\ \vdots \\ N_n(t) \end{bmatrix} \tag{9}$$

Let $\mathrm{N}(t)$ represent the number of atoms of each substance in the system at time t.

$$\mathrm{N}(t) = \begin{bmatrix} N_1(t) \\ N_2(t) \\ N_3(t) \\ \vdots \\ N_n(t) \end{bmatrix} \tag{10}$$

Therefore, substituting the $\mathrm{N}(t)$ vector into the left-hand side of the equation results in:

$$\frac{d}{dt} \begin{bmatrix} N_1(t) \\ N_2(t) \\ N_3(t) \\ \vdots \\ N_n(t) \end{bmatrix} = \frac{d\mathrm{N}(t)}{dt} \tag{11}$$

which may also be written as:

$$\mathrm{N}'(t) \tag{12}$$

Next, we can separate the right-hand side of the equation as the difference of two vectors.

$$\begin{bmatrix} \sum_{i=1}^{n} p_{i\to 1}\lambda_i N_i(t) \\ \sum_{i=1}^{n} p_{i\to 2}\lambda_i N_i(t) \\ \sum_{i=1}^{n} p_{i\to 3}\lambda_i N_i(t) \\ \vdots \\ \sum_{i=1}^{n} p_{i\to n}\lambda_i N_i(t) \end{bmatrix} - \begin{bmatrix} \lambda_1 N_1(t) \\ \lambda_2 N_2(t) \\ \lambda_3 N_3(t) \\ \vdots \\ \lambda_n N_n(t) \end{bmatrix} \tag{13}$$

To proceed, the two column vectors on the right-hand side of the equation are assessed individually.

First, consider the vector

$$\begin{bmatrix} \sum_{i=1}^{n} p_{i\to 1}\lambda_i N_i(t) \\ \sum_{i=1}^{n} p_{i\to 2}\lambda_i N_i(t) \\ \sum_{i=1}^{n} p_{i\to 3}\lambda_i N_i(t) \\ \vdots \\ \sum_{i=1}^{n} p_{i\to n}\lambda_i N_i(t) \end{bmatrix} \tag{14}$$

Recalling that…

$$\sum_{i=1}^{n} p_{i\to j}\lambda_i N_i(t) = p_{1\to j}\lambda_1 N_1(t) + p_{2\to j}\lambda_2 N_2(t) + p_{3\to j}\lambda_3 N_3(t) + \cdots + p_{n\to j}\lambda_n N_n(t)$$

We can express this vector as:

$$\begin{bmatrix} p_{1\to 1}\lambda_1 N_1(t) + p_{2\to 1}\lambda_2 N_2(t) + p_{3\to 1}\lambda_3 N_3(t) + & & +p_{n\to 1}\lambda_n N_n(t) \\ p_{1\to 2}\lambda_1 N_1(t) + p_{2\to 2}\lambda_2 N_2(t) + p_{3\to 2}\lambda_3 N_3(t) + & \cdots & +p_{n\to 2}\lambda_n N_n(t) \\ p_{1\to 3}\lambda_1 N_1(t) + p_{2\to 3}\lambda_2 N_2(t) + p_{3\to 3}\lambda_3 N_3(t) + & & +p_{n\to 3}\lambda_n N_n(t) \\ \vdots & \ddots & \vdots \\ p_{1\to n}\lambda_1 N_1(t) + p_{2\to n}\lambda_2 N_2(t) + p_{3\to n}\lambda_3 N_3(t) + & \cdots & +p_{n\to n}\lambda_n N_n(t) \end{bmatrix} \quad (15)$$

Noticing that each entry of this vector displays the same structure $p_{i\to j}, \lambda_i N_i(t)$, we can rewrite this vector as the product of a matrix and a vector. To achieve the matrix-vector expression, the decay probabilities and radioactive decay constants are separated from $\mathrm{N}(t)$:

$$\begin{bmatrix} p_{1\to 1}\lambda_1 & p_{2\to 1}\lambda_2 & p_{3\to 1}\lambda_3 & & p_{n\to 1}\lambda_n \\ p_{1\to 2}\lambda_1 & p_{2\to 2}\lambda_2 & p_{3\to 2}\lambda_3 & \cdots & p_{n\to 2}\lambda_n \\ p_{1\to 3}\lambda_1 & p_{2\to 3}\lambda_2 & p_{3\to 3}\lambda_3 & & p_{n\to 3}\lambda_n \\ & \vdots & & \ddots & \vdots \\ p_{1\to n}\lambda_1 & p_{2\to n}\lambda_2 & p_{3\to n}\lambda_3 & \cdots & p_{n\to n}\lambda_n \end{bmatrix} \times \begin{bmatrix} N_1(t) \\ N_2(t) \\ N_3(t) \\ \vdots \\ N_n(t) \end{bmatrix} \quad (16)$$

Recalling that…

$$\mathrm{N}(t) = \begin{bmatrix} N_1(t) \\ N_2(t) \\ N_3(t) \\ \vdots \\ N_n(t) \end{bmatrix}$$

Thus, substituting the $\mathrm{N}(t)$ vector into this expression results in:

$$\begin{bmatrix} p_{1\to1}\lambda_1 & p_{2\to1}\lambda_2 & p_{3\to1}\lambda_3 & & p_{n\to1}\lambda_n \\ p_{1\to2}\lambda_1 & p_{2\to2}\lambda_2 & p_{3\to2}\lambda_3 & \cdots & p_{n\to2}\lambda_n \\ p_{1\to3}\lambda_1 & p_{2\to3}\lambda_2 & p_{3\to3}\lambda_3 & & p_{n\to3}\lambda_n \\ & \vdots & & \ddots & \vdots \\ p_{1\to n}\lambda_1 & p_{2\to n}\lambda_2 & p_{3\to n}\lambda_3 & \cdots & p_{n\to n}\lambda_n \end{bmatrix} \mathrm{N}(t) \qquad (17)$$

Next, consider the remaining vector:

$$\begin{bmatrix} \lambda_1 N_1(t) \\ \lambda_2 N_2(t) \\ \lambda_3 N_3(t) \\ \vdots \\ \lambda_n N_n(t) \end{bmatrix} \qquad (18)$$

This column vector consists of the products of λ_j and $N_j(t)$. We can observe the same structure when we multiply a diagonal matrix with a column vector. Therefore, we can express this vector as a product of a diagonal matrix, consisting of λ_j, and a vector, consisting of $N_j(t)$:

$$\begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ & & & & \lambda_n \end{bmatrix} \times \begin{bmatrix} N_1(t) \\ N_2(t) \\ N_3(t) \\ \vdots \\ N_n(t) \end{bmatrix} \tag{19}$$

Substituting $\mathrm{N}(t)$ into this expression gives:

$$\begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ & & & & \lambda_n \end{bmatrix} \mathrm{N}(t) \tag{20}$$

Now that we have individually assessed both column vectors on the right-hand side of the equation, we can substitute the obtained representations of these vectors into the equation to form a more compact matrix notation:

$$\mathrm{N}'(t) = \begin{bmatrix} p_{1\to1}\lambda_1 & p_{2\to1}\lambda_2 & p_{3\to1}\lambda_3 & & p_{n\to1}\lambda_n \\ p_{1\to2}\lambda_1 & p_{2\to2}\lambda_2 & p_{3\to2}\lambda_3 & \cdots & p_{n\to2}\lambda_n \\ p_{1\to3}\lambda_1 & p_{2\to3}\lambda_2 & p_{3\to3}\lambda_3 & & p_{n\to3}\lambda_n \\ & \vdots & & \ddots & \vdots \\ p_{1\to n}\lambda_1 & p_{2\to n}\lambda_2 & p_{3\to n}\lambda_3 & \cdots & p_{n\to n}\lambda_n \end{bmatrix} \mathrm{N}(t) - \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ & & & & \lambda_n \end{bmatrix} \mathrm{N}(t) \tag{21}$$

Furthermore, we can rearrange this equation to obtain a single matrix acting on $\mathrm{N}(t)$ vector:

$$\mathrm{N}'(t) = \left(\begin{bmatrix} p_{1\to1}\lambda_1 & p_{2\to1}\lambda_2 & p_{3\to1}\lambda_3 & & p_{n\to1}\lambda_n \\ p_{1\to2}\lambda_1 & p_{2\to2}\lambda_2 & p_{3\to2}\lambda_3 & \cdots & p_{n\to2}\lambda_n \\ p_{1\to3}\lambda_1 & p_{2\to3}\lambda_2 & p_{3\to3}\lambda_3 & & p_{n\to3}\lambda_n \\ & \vdots & & \ddots & \vdots \\ p_{1\to n}\lambda_1 & p_{2\to n}\lambda_2 & p_{3\to n}\lambda_3 & \cdots & p_{n\to n}\lambda_n \end{bmatrix} - \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ & & & & \lambda_n \end{bmatrix} \right) \mathrm{N}(t) \tag{22}$$

Let K denote the matrix:

$$K = \begin{bmatrix} p_{1\to1}\lambda_1 - \lambda_1 & p_{2\to1}\lambda_2 & p_{3\to1}\lambda_3 & \cdots & p_{n\to1}\lambda_n \\ p_{1\to2}\lambda_1 & p_{2\to2}\lambda_2 - \lambda_2 & p_{3\to2}\lambda_3 & \cdots & p_{n\to2}\lambda_n \\ p_{1\to3}\lambda_1 & p_{2\to3}\lambda_2 & p_{3\to3}\lambda_3 - \lambda_3 & \cdots & p_{n\to3}\lambda_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{1\to n}\lambda_1 & p_{2\to n}\lambda_2 & p_{3\to n}\lambda_3 & \cdots & p_{n\to n}\lambda_n - \lambda_n \end{bmatrix} \quad (23)$$

Therefore, the equation can be expressed as:

$$\mathrm{N}'(t) = K\mathrm{N}(t) \quad (24)$$

Now, we express the rate of change of entire generalized system as a vector which is a product of matrix-vector multiplication. This compact formula includes every possible branch, isotope, decay that can be found in the system with their numerical data.

C. Analytical Solution of the Generalized Radioactive Decay System

In this section, we obtain the analytical solution for the vector that represents abundances of substances at time t. As mentioned in Section IV-A, Laplace transform is a powerful tool for solving first-order differential equations. Thus, we use Laplace transform and, later, inverse Laplace transform to determine the final expression for our system.

To start, we apply the Laplace transform to the differential equation derived in the previous section, (24):

$$\mathcal{L}(\mathrm{N}';s) = \mathcal{L}(K\mathrm{N};\mathrm{s}) \tag{25}$$

Recalling that…

$$\mathcal{L}(f;s) = \int_0^\infty e^{-st} f(t)dt$$

Therefore, we can express this equation as:

$$\int_0^\infty e^{-st}\mathrm{N}'(t)dt = \int_0^\infty e^{-st} K\mathrm{N}(t)dt \tag{26}$$

First, consider the integral on the left-hand side of the equation:

$$\int_0^\infty e^{-st}\mathrm{N}'(t)dt \tag{27}$$

By using integration by parts, we obtain:

$$\int_0^\infty e^{-st}\mathrm{N}'(t)dt = \left[e^{-st}\mathrm{N}(t)\right]_0^\infty - \int_0^\infty -se^{-st}\mathrm{N}(t)dt \tag{28}$$

Since $\Re(s) > 0$ and $\mathrm{N}(t)$ is bounded for all t,

$$\lim_{t\to\infty} e^{-st}\mathrm{N}(t) = 0 \tag{29}$$

We can rewrite the equation as:

$$\int_0^\infty e^{-st}\mathrm{N}'(t)dt = s\int_0^\infty e^{-st}\mathrm{N}(t)dt - \mathrm{N}(0) \tag{30}$$

Notice that we were able to express $\mathcal{L}(\mathrm{N}';s)$ in terms of $\mathcal{L}(\mathrm{N};s)$. Thus, we can substitute this expression (30) into the original equation (26) in order to solve it algebraically:

$$s\int_0^\infty e^{-st}\mathrm{N}(t)dt - \mathrm{N}(0) = \int_0^\infty e^{-st}\,K\mathrm{N}(t)dt \tag{31}$$

The integral on the right-hand side of the equation can also be expressed in terms of $\mathcal{L}(\mathrm{N}; s)$ by factoring the K matrix, and as K is constant, we can rewrite this equation as:

$$s\int_0^\infty e^{-st}\mathrm{N}(t)dt - \mathrm{N}(0) = K\int_0^\infty e^{-st}\,\mathrm{N}(t)dt \tag{32}$$

Substituting $\mathcal{L}(\mathrm{N}; s)$ into this equation results in:

$$s\mathcal{L}(\mathrm{N}; s) - \mathrm{N}(0) = K\mathcal{L}(\mathrm{N}; \mathrm{s}) \tag{33}$$

Now, we can rearrange the equation to isolate $\mathcal{L}(\mathrm{N}; s)$:

$$s\mathcal{L}(\mathrm{N}; \mathrm{s}) - K\mathcal{L}(\mathrm{N}; s) = \mathrm{N}(0) \tag{34}$$

Factoring out $\mathcal{L}(\mathrm{N}; s)$ in this equation gives:

$$(sI - K)\mathcal{L}(\mathrm{N}; s) = \mathrm{N}(0) \tag{35}$$

where I is an $n \times n$ identity matrix.

To isolate $\mathcal{L}(\mathrm{N};s)$, assuming that $(sI-K)$ is invertible, the equation must be multiplied by $(sI-K)^{-1}$,

$$(sI-K)^{-1}(sI-K)\mathcal{L}(\mathrm{N};s)=(sI-K)^{-1}\mathrm{N}(0) \tag{36}$$

Which simplifies to:

$$\mathcal{L}(\mathrm{N};s)=(sI-K)^{-1}\mathrm{N}(0) \tag{37}$$

At this point, we have algebraically solved the equation in the Laplace domain. Hence, we must apply the inverse Laplace transform to convert our equation back to the time domain:

$$\mathcal{L}^{-1}\big(\mathcal{L}(\mathrm{N};s)\big)=\mathcal{L}^{-1}\big((sI-K)^{-1}\mathrm{N}(0)\big) \tag{38}$$

Simplifying the left-hand side of the equation gives:

$$\mathrm{N}(t)=\mathcal{L}^{-1}\big((sI-K)^{-1}\mathrm{N}(0)\big) \tag{39}$$

Now, consider the right-hand side of the equation. In order to obtain solutions for this expression, we represent the inverse Laplace transform as a Bromwich integral, using Mellin's inversion formula.

Recalling that…

$$\mathcal{L}^{-1}\{\mathcal{L}(f;s)\}(t) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT} e^{st}\mathcal{L}(f;s)ds$$

where $\gamma \in \mathbb{R}$ and $\Re(\gamma) > \Re(a_k)$ for all k, with a_k representing the k^{th} pole of $\mathcal{L}(f;s)$. Hence, to apply Mellin's inversion formula, we need to determine an appropriate value for γ.

Let $\gamma \in \mathbb{R}$ and $\Re(\gamma) > \Re(a_k)$ for all k, with a_k representing the k^{th} pole of $(sI - K)^{-1}$. Therefore, we can apply Mellin's inversion formula to the expression on the right-hand side of the equation:

$$\mathrm{N}(t) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT} e^{st}(sI - K)^{-1}\mathrm{N}(0)ds \qquad (40)$$

Next, we factor out the vector $\mathrm{N}(0)$ from the integral on the right-hand side of the equation, as $\mathrm{N}(0)$ is a constant column vector, to simplify it:

$$\mathrm{N}(t) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT} e^{st}(sI - K)^{-1}ds\,\mathrm{N}(0) \qquad (41)$$

To compute this integral, we must use Cauchy's residue theorem, which requires a closed contour in order to operate. However, our Bromwich integral is a line integral and does not form a closed contour. This can be clearly seen in figure, 2:

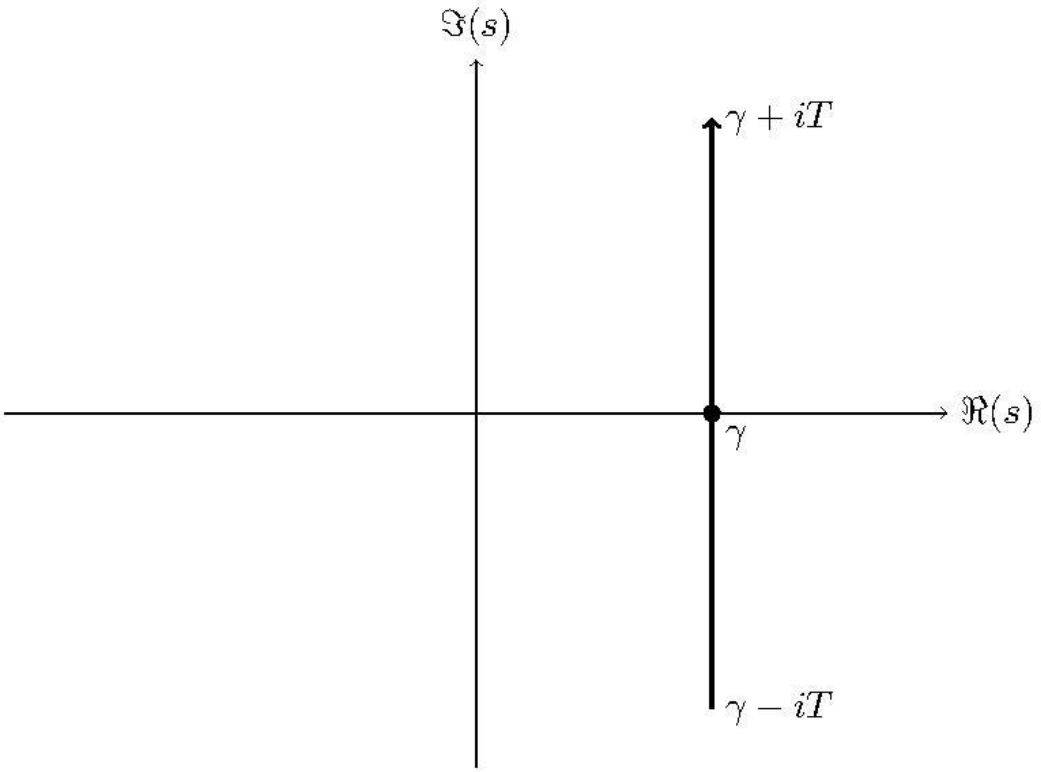

Fig. 2. Bromwich integral shown in an Argand diagram (code provided in Appendix A) [18] [19]

To resolve this problem, we can add an arc (a smooth curve) that connects $\gamma + iT$ and $\gamma - iT$.

Let Γarc refer to the arc that connects $\gamma + iT$ and $\gamma - iT$, which is parametrized by

$$s(\theta) = \gamma + Te^{i\theta} \tag{42}$$

[18] "Overleaf LaTeX code editor" overleaf.com. https://tr.overleaf.com (accessed December 14, 2025)

[19] "ChatGPT: smart and simple AI" chatgpt.com. https://chatgpt.com (accessed December 14,2025)

where $\theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$. Hence, integrating our Bromwich integral's integrand, $e^{st}(sI - K)^{-1}$, along Γarc gives:

$$\int_{\Gamma arc} e^{st}(sI - K)^{-1} ds \tag{43}$$

Thus, a closed contour is formed by joining the arc and the Bromwich integral, which is shown in figure, 3:

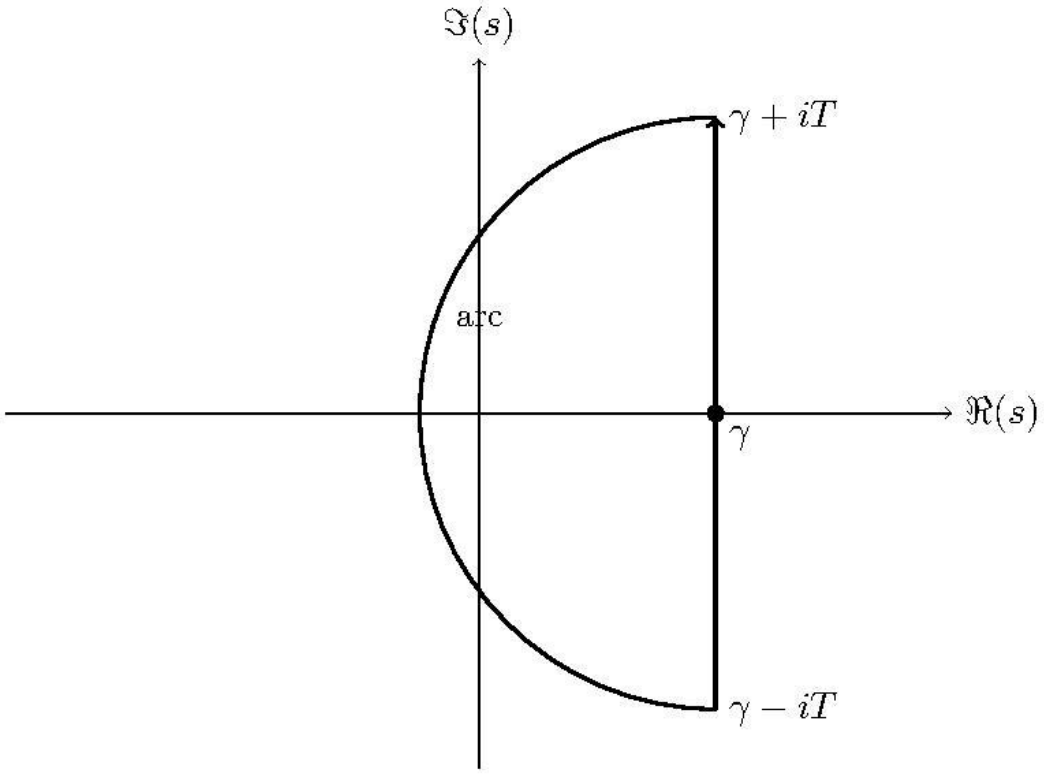

Fig. 3. Closed contour formed on the complex plane by connecting the arc with the Bromwich integral (code provided in Appendix B) [18] [19]

[18] "Overleaf LaTeX code editor" overleaf.com. https://tr.overleaf.com (accessed December 14, 2025)

[19] "ChatGPT: smart and simple AI" chatgpt.com. https://chatgpt.com (accessed December 14,2025)

Let Γ denote the closed contour formed by the arc and the Bromwich integral. Knowing that Γ consists of a line segment and an arc, we can express the contour integral of Γ as the sum of our Bromwich integral and the Γarc integral:

$$\int_{\gamma-iT}^{\gamma+iT} e^{st}(sI-K)^{-1}ds + \int_{\Gamma arc} e^{st}(sI-K)^{-1}ds = \oint_{\Gamma} e^{st}(sI-K)^{-1}ds \qquad (44)$$

Next, we must isolate the Bromwich integral as it is the integral that we are trying to compute:

$$\int_{\gamma-iT}^{\gamma+iT} e^{st}(sI-K)^{-1}ds = \oint_{\Gamma} e^{st}(sI-K)^{-1}ds - \int_{\Gamma arc} e^{st}(sI-K)^{-1}ds \qquad (45)$$

Now that we have expressed our Bromwich integral as a contour integral and an arc integral, we can substitute this equation into our original equation (41) so that we can apply Cauchy's residue theorem to obtain the solutions.

$$\mathrm{N}(t) = \frac{1}{2\pi i}\lim_{T\to\infty}\left(\oint_{\Gamma} e^{st}(sI-K)^{-1}ds - \int_{\Gamma arc} e^{st}(sI-K)^{-1}ds\right)\mathrm{N}(0) \qquad (46)$$

First, consider the contour integral, $\oint_{\Gamma} e^{st}(sI-K)^{-1}ds$.

Recalling that…

$$\oint_C f(s)\,ds = 2\pi i \sum \text{Res}(f, a_k)$$

where a_k refers to the k^{th} pole of f inside C.

Therefore, we can express our contour integral as:

$$\oint_\Gamma e^{st}(sI - K)^{-1} ds = 2\pi i \sum_{k=1}^{n} \text{Res}(e^{st}(sI - K)^{-1}, a_k) \tag{47}$$

To proceed, we must determine the corresponding values of a_k, the poles of $e^{st}(sI - K)^{-1}$. For this function, e^{st} does not have poles because it is not singular at any point. However, since the existence of $(sI - K)^{-1}$ is dependent on the invertibility of $(sI - K)$, $(sI - K)^{-1}$ can be singular when $\det(sI - K) = 0$. Thus, the poles of our function are the values of s where $\det(sI - K) = 0$.

Let μ_k be the k^{th} eigenvalue of matrix K. According to the definition of eigenvalue,

$$\det(\mu_k I - K) = 0 \tag{48}$$

Noticing that the equation $\det(sI - K) = 0$ is the same equation that is used to define the μ_k, we determine that the poles of our function occur when $s = \mu_k$.

Hence, the corresponding values of a_k are μ_k, the eigenvalues of matrix K. Now, we can substitute μ_k into our contour integral equation (47) in order to obtain:

$$\oint_{\Gamma} e^{st}(sI-K)^{-1}ds = 2\pi i \sum_{k=1}^{n} \text{Res}(e^{st}(sI-K)^{-1}, \mu_k) \tag{49}$$

Recalling that…

$$\text{Res}(f,\varphi) = \frac{1}{(m-1)!}\lim_{s\to\varphi}\frac{d^{m-1}}{ds^{m-1}}\big((s-\varphi)^m f(s)\big)$$

Where m represents the order of φ.

Therefore, the residue of our function can be expressed as:

$$\text{Res}(e^{st}(sI-K)^{-1}, \mu_k) = \frac{1}{(m_k-1)!}\lim_{s\to\mu_k}\frac{d^{m_k-1}}{ds^{m_k-1}}((s-\mu_k)^{m_k}e^{st}(sI-K)^{-1}) \tag{50}$$

Substituting this expression into our contour integral equation (50) gives:

$$\oint_{\Gamma} e^{st}(sI-K)^{-1}ds = 2\pi i \sum_{k=1}^{n} \frac{1}{(m_k-1)!}\lim_{s\to\mu_k}\frac{d^{m_k-1}}{ds^{m_k-1}}((s-\mu_k)^{m_k}e^{st}(sI-K)^{-1}) \tag{51}$$

At this point, we have applied Cauchy's residue theorem to our contour integral. Thus, we substitute this expression (51) into our original equation (46) to obtain the final expression for $\mathrm{N}(t)$:

$$\mathrm{N}(t) = \sum_{k=1}^{n} \frac{1}{(m_k - 1)!} \lim_{s \to \mu_k} \frac{d^{m_k - 1}}{ds^{m_k - 1}} ((s - \mu_k)^{m_k} e^{st} (sI - K)^{-1}) \, \mathrm{N}(0) - \frac{1}{2\pi i} \lim_{T \to \infty} \int_{\Gamma arc} e^{st} (sI - K)^{-1} ds \, \mathrm{N}(0) \qquad (52)$$

Next, consider the path integral of Γarc:

$$\int_{\Gamma arc} e^{st} (sI - K)^{-1} ds \qquad (53)$$

If we substitute the parametrized expression of Γarc, $s(\theta) = \gamma + Te^{i\theta}$, into this integral, we can obtain:

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{\, s(\theta)t} (\, s(\theta)I - K)^{-1} iTe^{i\theta} d\theta \qquad (54)$$

Using the $e^{i\theta} = \cos(\theta) - i\sin(\theta)$ identity, we can rewrite $s(\theta)$ as

$$s(\theta) = \gamma + T(\cos(\theta) - isin(\theta)) \qquad (55)$$

Thus, substituting this expression into the path integral of Γarc gives:

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{[\gamma+T(\cos(\theta)-i\sin(\theta))]t}\left(\left[\gamma+T\left(\cos(\theta)-i\sin(\theta)\right)\right]I-K\right)^{-1}iTe^{i\theta}d\theta \tag{56}$$

Knowing that $\cos(\theta) \leq 0$, as $\theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, we can evaluate the following limit,

$$\lim_{T\to\infty}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{[\gamma+T(\cos(\theta)-i\sin(\theta))]t}\left(\left[\gamma+T\left(\cos(\theta)-i\sin(\theta)\right)\right]I-K\right)^{-1}iTe^{i\theta}d\theta \tag{57}$$

By bounding its absolute value, following the standard estimation method for complex integrals in complex analysis:

$$\left|\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{[\gamma+T(\cos(\theta)-i\sin(\theta))]t}\left(\left[\gamma+T\left(\cos(\theta)-i\sin(\theta)\right)\right]I-K\right)^{-1}iTe^{i\theta}d\theta\right| \tag{58}$$

Which simplifies to:

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{\gamma t + T\cos(\theta)t} \left| \left(\left[\gamma + T\left(\cos(\theta) - i\sin(\theta)\right)\right] I - K \right)^{-1} \right| T d\theta \qquad (59)$$

At this point, we know that $|s(\theta)| = \left|\gamma + T\left(\cos(\theta) - i\sin(\theta)\right)\right| \geq T - |\gamma|$. Thus, matrix $\left(\left[\gamma + T\left(\cos(\theta) - i\sin(\theta)\right)\right] I - K \right)$ is invertible, and $\left(\left[\gamma + T\left(\cos(\theta) - i\sin(\theta)\right)\right] I - K \right)^{-1} \approx \frac{1}{|s|}$ where T is sufficiently large. Therefore, $\left| \left(\left[\gamma + T\left(\cos(\theta) - i\sin(\theta)\right)\right] I - K \right)^{-1} \right| T$ remains bounded for all points on Γarc.

Now, using the fact that $\cos(\theta) \leq 0$, we conclude that as $T \to \infty$, the path integral of Γarc goes to 0 because $\lim_{T\to\infty} e^{\gamma t + T\cos(\theta)t} = 0$, as it dominates the $\left| \left(\left[\gamma + T\left(\cos(\theta) - i\sin(\theta)\right)\right] I - K \right)^{-1} \right| T$. Hence, the Γarc vanishes so we can express the $\mathrm{N}(t)$ as

$$\mathrm{N}(t) = \sum_{k=1}^{n} \frac{1}{(m_k - 1)!} \lim_{s\to\mu_k} \frac{d^{m_k-1}}{ds^{m_k-1}} \left[(s - \mu_k)^{m_k} e^{st} (sI - K)^{-1}\right] \mathrm{N}(0) \qquad (60)$$

At this point, we can enhance the computational capabilities of our expression (60) by applying the general Leibniz rule for derivative of a product. This way, during the computation of multiple time points, we can reuse the same derivative structure and only compute the time-dependent part. Thus, we can significantly increase both the computational efficiency and

speed of our model. General Leibniz rule states if we take the derivative of h and b function for τ-times [20], then,

$$(hb)^{(\tau)} = \sum_{q=0}^{\tau} \binom{\tau}{q} h^{(\tau-q)} b^{(q)}$$

Let $h(s) = e^{st}$, $b(s) = (s-\mu_k)^{m_k}(sI-K)^{-1}$, and $\tau = m_k - 1$. Thus, we can express the right-hand side of our equation (60) as

$$\sum_{k=1}^{n} \frac{1}{(m_k-1)!} \lim_{s\to\mu_k} \sum_{q=0}^{m_k-1} \binom{m_k-1}{q} \left[\frac{d^{m_k-1-q}}{ds^{m_k-1-q}}(e^{st})\right]\left[\frac{d^q}{ds^q}((s-\mu_k)^{m_k}(sI-K)^{-1})\right] \mathrm{N}(0) \qquad (61)$$

To simplify the equation, first, consider the term

$$\frac{d^{m_k-1-q}}{ds^{m_k-1-q}} e^{st} \qquad (62)$$

We know that $\frac{d^r}{ds^r} e^{st} = t^r e^{st}$ where $r \epsilon \mathbb{N}$. Thus,

[20] P. J. Olver, *Applications of Lie Groups to Differential Equations*, New York, NY, USA: Springer, 2000, pp. 318–319.

$$\frac{d^{m_k-1-q}}{ds^{m_k-1-q}} e^{st} = t^{m_k-1-q} e^{st} \tag{63}$$

Next, we consider:

$$\binom{m_k - 1}{q} \tag{64}$$

We can expand this combination as

$$\frac{(m_k - 1)!}{q!\,(m_k - 1 - q)!} \tag{65}$$

Now, we can substitute the terms (63) and (65) into our original equation (61) in order to obtain the final expression for our mathematical model:

$$\mathrm{N}(t) = \sum_{k=1}^{n} e^{\mu_k t} \sum_{q=0}^{m_k-1} \frac{1}{q!\,(m_k - q - 1)!} t^{m_k-q-1} \lim_{s\to\mu_k} \frac{d^q}{ds^q} ((s - \mu_k)^{m_k} (sI - K)^{-1})\,\mathrm{N}(0) \tag{66}$$

VI. Conclusion and Evaluation

This investigation aimed to develop a mathematical model for describing radioactive decay chains with branching probabilities by utilizing matrix algebra and Laplace transform. To generalize the model, Rutherford's formulation for radioactive decay was modified to

include a variable, $p_{i\to j}$, that represents the branching probabilities between the isotopes. This led to a system of several coupled first-order differential equations which are dependent on each other. In the following sections, this system was represented as vectors and matrices which allowed for easier algebraic manipulation of several coupled first-order differential equations [21]. Ultimately, constant matrix K was formed which helped to utilize the Laplace transform's linear properties to solve the system and express the solution in terms of the eigenvalues of K.

Matrix algebra utilized during the formation of the model enabled multiple advantages by allowing the entire decay chain to be expressed through a single square matrix K acting on the abundance vector $\mathrm{N}(t)$. K encodes all the coefficients found in the system in a concise manner and is not assumed to be a triangular matrix. Thus, the model can represent arbitrary chains, assuming that the entries satisfy the physically allowed decays. Additionally, this structure provides an advantage during the data entry stage in computation by not requiring the isotopes to be ordered by their position in the chain. Moreover, as the decay system is represented through the matrix K acting on the vector $\mathrm{N}(t)$, the formulation is valid for all $\mathrm{N}(0)$ rather than assuming that the initial abundances of substances other than N_1 are 0. These strengths of this mathematical model provide both practical and general advantages over the previous model, the Bateman Equation.

Laplace transform and Cauchy's residue theorem were used to obtain the final expression for the mathematical model, which allowed the solution to be expressed in terms of the eigenvalues of K. These eigenvalues represent the decay modes found in the chain. Thus, while computing multiple timepoints or different $\mathrm{N}(0)$ in a radioactive decay chain, eigenvalues can be

[21] L. Moral and A. F. Pacheco, "Algebraic approach to the radioactive decay equations," Am. J. Phys., vol. 71, no. 7, pp. 684–686, Jul. 2003, doi: 10.1119/1.1571834.

calculated once and reused for the other timepoints or initial conditions, $\mathrm{N}(0)$, providing computational efficiency.

However, the developed mathematical model has limitations that may impact the generality and accuracy. The model assumes that the system is a deterministic homogeneous closed system where no chemical reaction occurs and $p_{i\to j}\lambda_i$ is independent of time and environment. Thus, it cannot be reliably used to compute decay chains which do not fulfil these assumptions. Additionally, the vector $\mathrm{N}(t)$ does not track the emitted particles during the α and β decays resulting inaccuracies in specific systems which track the abundance of helium-4 isotopes. Moreover, the closed-form solution obtained from the Laplace transform is algebraically long and, for large decay networks, is not practical for hand calculation; in realistic applications it would need to be implemented numerically using standard eigenvalue algorithms or differential equation solvers rather than evaluated symbolically.

Possible extensions of this work include allowing the decay matrix K to depend on time or environmental parameters, adding source terms to model external production, and enlarging the state vector to include emitted particles and helium-4 explicitly, before comparing a numerical implementation of this model with existing Bateman-based computational approaches. Overall, matrix algebra was used to encode the full decay network in the matrix K, and the Laplace transform was used to obtain a reusable spectral solution for $\mathrm{N}(t)$, so together they provide a general framework for modelling radioactive decay chains with branching probabilities.

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APPENDIX A

CODE WRITTEN BY CHATGPT TO GENERATE THE BROMWICH INTEGRAL'S GRAPH

latex Copy code

```latex
\documentclass[tikz,border=5pt]{standalone}
\usepackage{tikz}

\begin{document}

\begin{tikzpicture}[scale=1.1]
  % parameters (drawing values)
  \def\gammaVal{2.0}
  \def\Tval{2.5}

  % axes
  \draw[->] (-4,0) -- (4,0) node[right] {$\Re(s)$};
  \draw[->] (0,-3) -- (0,3) node[above] {$\Im(s)$};

  % Bromwich line Re(s) = gamma
  \draw[thick,->] (\gammaVal,-\Tval) -- (\gammaVal,\Tval);

  % bold point at gamma on the real axis
  \fill (\gammaVal,0) circle (2pt);
  \node[below right] at (\gammaVal,0) {$\gamma$};

  % labels for endpoints gamma ± iT
  \node[right] at (\gammaVal,\Tval) {$\gamma + iT$};
  \node[right] at (\gammaVal,-\Tval) {$\gamma - iT$};

\end{tikzpicture}

\end{document}
```

APPENDIX B

CODE WRITTEN BY CHATGPT TO GENERATE THE CLOSED CONTOUR INTEGRAL'S GRAPH

latex Copy code

```latex
\documentclass[tikz,border=5pt]{standalone}
\usepackage{tikz}

\begin{document}

\begin{tikzpicture}[scale=1.1]
  % parameters
  \def\gammaVal{2.0}
  \def\Tval{2.5}

  % axes
  \draw[->] (-4,0) -- (4,0) node[right] {$\Re(s)$};
  \draw[->] (0,-3) -- (0,3) node[above] {$\Im(s)$};

  % Bromwich line
  \draw[thick,->] (\gammaVal,-\Tval) -- (\gammaVal,\Tval);

  % gamma point
  \fill (\gammaVal,0) circle (2pt);
  \node[below right] at (\gammaVal,0) {$\gamma$};

  % endpoints gamma ± iT
  \node[right] at (\gammaVal,\Tval) {$\gamma + iT$};
  \node[right] at (\gammaVal,-\Tval) {$\gamma - iT$};

  % semicircular arc closing to the left
  \draw[thick] (\gammaVal,\Tval) arc[start angle=90,end angle=270,radius=\Tval];

  % label for the arc
  \node at (\gammaVal-\Tval*0.8,0.8) {\small arc};

\end{tikzpicture}

\end{document}
```