Research Question:

"How does geometry relate to the solutions of cubic equations?"

3935 Words

Contents:

Abstract:

In this essay, I wanted to investigate the research question "How does geometry relate to the solutions of cubic equations?" These equations are basically polynomials and polynomials are mathematical expressions that solely use addition, subtraction, and multiplication to combine their variables and coefficients. The Greek words "poly" and "nomos" which both indicate "terms," are where the term "polynomial" originates. A polynomial contains constants, variables, and exponents. The degree of a polynomial is the highest exponential power of the variable in the expression.

Introduction:

This Extended Essay focuses on polynomials, which is a wide branch of Mathematics that consists of variables and coefficients. Polynomials were one of the topics that made Mathematicians think about for decades, in ancient times. Starting from the existence of people, the polynomials were everywhere in our lives. People used polynomials to find answers in their real-life situations. An example for this is finding the annual flow of River Nile. In ancient times, mathematicians used quadratic equations to find the annual flow of River Nile. Also, they were trying to find basic solutions for quadratic equations ($ax^2 + bx + c = 0$) and cubic equations ($ax^3 + bx^2 + cx + d = 0$), in order to use mathematics in their lives more easily.

In addition, since the time of the ancient Babylonians, Egyptians, and Greeks, polynomials have been studied and applied in various fields. Polynomials were employed by the Babylonians and Egyptians to approach real-world arithmetic and geometric issues. Polynomials were utilized by mathematicians in ancient Greece to learn algebra and number theory, including Euclid and Diophantus.

The solutions of first and second order polynomials had been found easily and they were able to relate these solutions to geometry. However, they could not find a solution for third order polynomials for years. They were focusing on this topic, but they were not able to find a solution for decades.

In this essay, the relation between geometry and third order polynomials will be investigated. I wanted to investigate this topic, because I wondered how I could easily solve third order polynomials. Solving second order polynomials easily made me think that there should be an easy solution for third order polynomials, too. I knew how I could solve second order polynomials, and geometric proof of them, but no matter how hard I tried, I could not find a solution to them. I tried to geometrically solve the third order polynomials multiple times, but I could not be able to find a proof on my own.

Background Information and Methodology:

Solution of Quadratic Equations

As solutions of second order polynomials had been known, which is $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$, there were also methods to use geometry to find the solution. The method to show the geometric solution of second order polynomials was to use geometrical objects that are equal to terms with the variable x. For an equation in form $ax^2 + bx = -c$, a square with lengths x (where $a = 1$) and 2 rectangles with lengths $\frac{b}{2a}$ and x will be used to geometrically prove the equation. These three objects will be combined to each other to form a square with lengths $x + \frac{b}{2a}$.

However, after combining these three objects, there will be a space in the bigger square. To fill this space and complete the square, another square with lengths $\frac{b}{2a}$ will be needed, and the area of this square will be added to both sides of the equation. As a result, there will be square with lengths $x + \frac{b}{2a}$ and equation in form of $\left(x + \frac{b}{2a}\right)^2 = -c + \frac{b^2}{4a^2}$ will be obtained. To solve the equation, one side of the equation will be equaled to zero and equation in form of $x^2 + \frac{bx}{a} + \frac{c}{a}$ will be obtained. Finally, factorizing the equation, which will occur one solution (as the final equation is a squared term) and the solution of the quadratic polynomial would be done.

An example for this method is the proof of the equation $x^2 + 26x = 27$. In this situation, a square with the length of x and 2 rectangles with the area of $13x$ will be needed. After combining these three objects, the square with lengths $x + 13$ will miss an area in it, which is equal to 169. After adding 169 to both sides of the equation, the equation

$(x + 13)^2 = 196$ will be obtained. As a final step, equalizing the sides of the equation, $x = 1$ will be obtained as a solution of the equation. This method of geometrically proving the quadratic polynomials led other mathematicians to find a way to solve third order polynomials in the future.

3. Geometric Proof to a Quadratic Equation

General Quadratic Equation:

$$ax^2 + bx + c = 0$$

One can divide both sides with a and represent any quadratic formula as following.

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

As it can be seen from the figure:

$$(x + \frac{b}{2a})^2 = \frac{b^2}{4a^2} - \frac{c}{a} \Rightarrow x = -\frac{b}{2a} \pm \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}$$

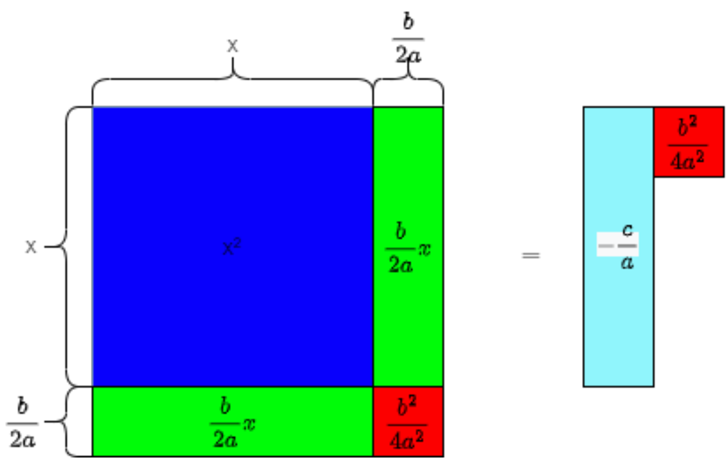

Figure 1: Geometrical Proof of A Quadratic Polynomial which has an equation:

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}+\frac{c}{a}$$

Let's use this method to find the roots of a quadratic equation.

Above *Methodology* will be used:

$$x^2+26x-27=0$$

For this example, at first, to avoid the negative areas, we need to change the equation into:

$$x^2+26x=27$$

After that, to find a solution geometrically, we need to use a square with lengths x cm, 2 rectangles with areas 13 cm and x cm are needed. After combining these, to form an uncompleted square with lengths $x+13$, a square with area $169\ cm^2$, which will turn the equation into following:

$$x^2+26x+169=27+169 \Rightarrow x^2+26x+169=196$$

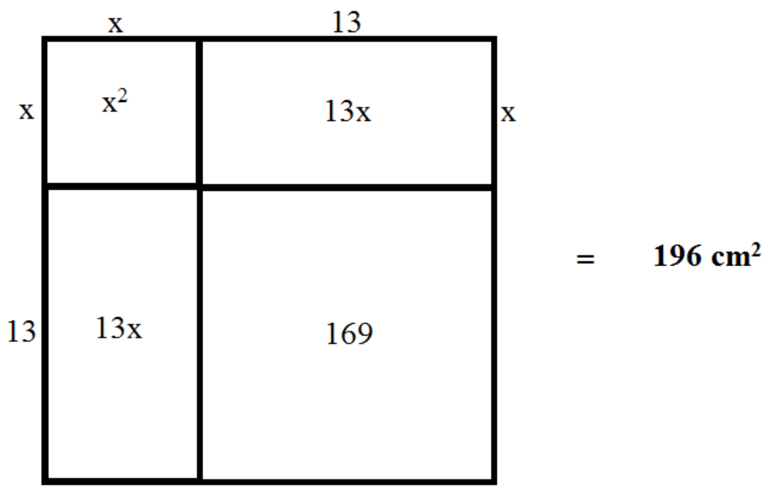

Figure 2: Geometrical Representation of the equation

Solution of Depressed Cubic

Luca Pacioli, who was a famous and esteemed mathematician of his time, had written a book about all topics of mathematics. He gave third order polynomials a place in his book, and he had written that the solutions of third order polynomials are impossible. As Pacioli claimed in his book it is believed that finding a solution to any third order polynomial is impossible, as none of the famous mathematicians could find a solution for third order polynomials before, until Scipione Del Ferro had found a method to solve a depressed cubic, which are in form of $ax^3 + bx + c = 0$. As Del Ferro is the one and only person to find a solution to cubic equations, he did not want to lose his prestige and his title as mathematician, and he did not reveal his method. Del Ferro did not tell his method until his deathbed. He revealed his method in his deathbed to his student Antionio Fior. Despite his teacher, he used the method to challenge another respected mathematician to a math duel: Niccolo Fontana Tartaglia.

In math duels, each of the mathematicians asks 30 questions to each other and they have 40 days to solve these questions. Fior gives 30 questions of depressed cubic to Tartaglia. Tartaglia manages to solve all of the questions in 2 hours, while Fior could not solve a single problem. Before the duel, Tartaglia learns that Fior has a method to solve a depressed cubic, and he understands that a mathematician revealed his method to Fior. After that, Tartaglia tries to find a solution for depressed cubic equations. To do it, he uses geometry, and he extends the method of completing the square in three dimensions.

Tartaglia uses a depressed cubic equation in form of $ax^3 + bx = c$ and uses a cube with lengths x and he thinks about extending this cube in all dimensions with length y, and equalize its' lengths to z. As a result, he obtains a cube in form of $(x + y)^3$, which is equal to z^3. He then divides this cube to 7 parts. Three of the parts have volume of $x \times x \times y$, three of them have volume of $x \times y \times y$ and 2 cubes with volumes $x^3 and\ y^3$. After combining 6 rectangles to each other, with lengths $3y, x, z$, he obtains that the volume of this object is equal to bx in the main equation. After this, the cube with the volume of z^3 could be obtained in equation, with adding y^3 to both sides of the equation. At final, using the equation$ax^3 + bx + y^3 = c + y^3$and writing the known information into to formula, Tartaglia finds the solution of a depressed cubic polynomial.

$$a\left(x-\frac{b}{3a}\right)^3+b\left(x-\frac{b}{3a}\right)^2+c\left(x-\frac{b}{3a}\right)+d=0$$

$$ax^3\boxed{\cancel{-\frac{2bx^2}{3}}}+\frac{b^2x}{9}\boxed{\cancel{-\frac{bx^2}{3}}}+\frac{2b^2x}{3}-\frac{b^3}{27a}\boxed{\cancel{+bx^2}}-\frac{2b^2x}{3a}+\frac{b^3}{9a^2}\cdots=0$$

$$ax^3+\left(c-\frac{b^2}{3a}\right)x+\left(d+\frac{2b^3}{27a^2}-\frac{bc}{3a}\right)=0$$

Figure 3: Tartaglia's Method of Changing Cubic Polynomials to Depressed Cubics

To find the roots of a depressed cubic equation, similar method that had been used in quadratic equations should be used:

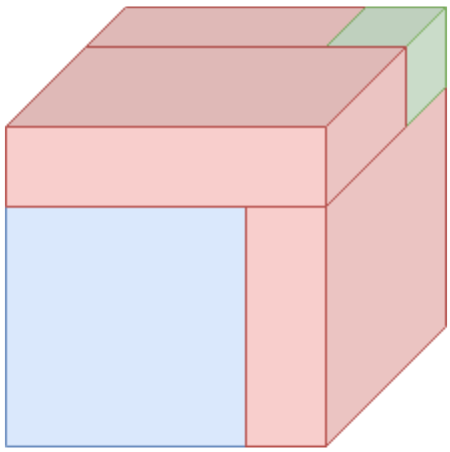

Figure 4: A Cube with volume $z^3 = (x + y)^3$ which is separated to use the method, which is used in solving a depressed cubic equation

Depressed Cubic equations have the form:

$$ax^3 + bx + c = 0 \ \ or \ x^3 + px = q$$

Let the smaller cube in the figure denote x^3 and greater cube denote $(x + y)^3$ for a y. Then, the red rectangular prisms correspond to a total volume of $3xy(x + y)$ and the green cube correspond to a volume of y^3. Then, one can equate $3xy(x + y) = px$, add y^3 on both sides of the equation to get:

$$(x + y)^3 = y^3 + q$$

Since,

$$3xy(x + y) = px$$

$$(x + y) = \frac{p}{3y}$$

Putting in the equation one gets the following:

$$p^3 = 27y^6 + 27qy^3$$

Now letting $u = y^3$ and using the quadratic formula:

$$p^3 = 27u^2 + 27qu$$

$$u = -\frac{27q}{54} \pm \sqrt{\frac{3^6 \times q^2}{4 \times 3^6} + \frac{p^3}{27}}$$

NOTE: Here, $\pm$ sign is used for the sake of mathematical notation. However, in Ancient times, $\pm$ did not exist, so in future numerical solutions, instead of $\pm$ sign, the sign which makes the equation positive will be used, which are $(+)$ or $(-)$.

$$p = -6 \; and \;\; q = 2$$

$$y = \sqrt[3]{-\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

Therefore

$$x = \frac{p}{3\left(\sqrt[3]{-\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}\right)} - \sqrt[3]{-\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

Let's use the methodology to find the solution of a depressed cubic, using the steps above

$$x^3 + 48x - 504 = 0$$

To solve this Depressed Cubic, first the equation will be changed into,

as $p = 48$ and $q = 504$ for this situation:

$$x^3 + 48x = 504$$

After using steps above; equation below will be obtained:

$$3 \times x \times y \times (x + y) = 48x$$

Then, x will be deleted from both sides of the equation:

$$3 \times y \times (x + y) = 48$$

Then,

$$(x+y)=\frac{48}{3y}=\frac{16}{y}$$

After combining these with, following equation will be obtained:

$$48^3 = 27y^6 + 27\times 504\times y^3$$

Let $y^3 = u$,

$$110592 = 27\times u^2 + 27\times 504\times u$$

$$u=\frac{-27\times 504}{54}+\sqrt{\frac{3^6\times 504^2}{4\times 3^6}+\frac{48^3}{27}}$$

Which is:

$$u=-252+\sqrt{63504+4096}$$

After adding these number, the value of u will be obtained:

$$u=8$$

Retransforming $u\ to\ y$:

$$y=\sqrt[3]{8}\Rightarrow y=2$$

Finally, the solution would be found like following:

$$x=\frac{48}{3\times 2}-2$$

$$\Rightarrow x=6$$

Finding a Solution to Cubic Equations

As mentioned throughout the essay, the geometric solution of cubic roots is not totally a creation of one mathematician, but combination of some findings of different mathematicians. In order to solve any arrangement of cubic equations, using geometry, Cardano's Method of changing cubic equations to depressed cubic equations is the first method to use. His method was trying to find a solution of a cubic equation, which is in the form of $ax^3 + bx^2 + cx + d$, he realizes that substituting for x, $x - \frac{b}{3a}$ will cancel out all the x^2 terms in the equation, which leads to obtain a depressed cubic equation.

After he finds the method, he writes is as a poem, but like Del Ferro, he also refuses to reveal his method. He keeps his method as a secret, until a persistent mathematician named Gerolamo Cardano desperately writes to Tartaglia multiple times and shows how desire he is to learn the method. Eventually, Tartaglia reveals his method to Cardano, on one condition: Tartaglia makes Cardano to promise that he will not reveal the method to another person. As Cardano learns the method to solve depressed cubic equations, he tries to find a solution to all other arrangements of cubic equations.

While working on this, Cardano finds a method to change any general cubic equation to a depressed cubic equation. Cardano realizes that substituting for x, $x - \frac{b}{3a}$ will cancel out all the x squared terms in the equation, which was in form of $ax^3 + bx^2 + cx + d = 0$. By combining his and Tartaglia's method, Cardano solves any cubic polynomials. Despite others, Cardano does not wants to be known as a mathematician and he wants to reveal the solution of any cubic equation. However, as he promised to Tartaglia about not revealing his method, he wants to learn another method to solve a depressed cubic. Cardano visits son-inlaw of Del Ferro, and he learns the method of Del Ferro to solve a depressed cubic equation. After combining his and Del Ferro's methods, Cardano reveals a book about Mathematics, which is named "ARTIS MAGNÆ", and he includes solutions of all arrangements of cubic equations in his book.

So, the solution for a cubic equation will first follow the steps below, then steps that had been used for depressed cubic equations will be used, as solutions of cubic equations are made up from transforming cubic equations to depressed cubic equations and then solving them.

To solve any arrangement of cubic equations, Cardano's Method of changing cubic equations to depressed cubic should be used first. His method was trying to find a solution of a cubic equation, which is in the form of $ax^3 + bx^2 + cx + d$, he realizes that substituting for x, $x - \frac{b}{3a}$ will cancel out all the x^2 terms in the equation, which leads to obtain a depressed cubic equation. The steps are shown below:

$$ax^3 + bx^2 + cx + d = 0 \tag{1}$$

$$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0 \tag{2}$$

Letting $u - \frac{b}{3a} = x$ one gets:

$$\left(u - \frac{b}{3a}\right)^3 + \frac{b}{a}\left(u - \frac{b}{3a}\right)^2 + \frac{c}{a}\left(u - \frac{b}{3a}\right) + \frac{d}{a} = 0 \tag{3}$$

$$u^3 + \left(\frac{c}{a} - \frac{b^2}{3a^2}\right)u = \left(\frac{bc}{3a^2} - \frac{2b^3}{27a^3} - \frac{d}{a}\right) \tag{4}$$

Equation (4) above shows that any cubic polynomial can be expressed in terms of a depressed cubic. One can replace $p = \left(\frac{c}{a} - \frac{b^2}{3a^2}\right)$ and $q = \left(\frac{bc}{3a^2} - \frac{2b^3}{27a^3} - \frac{d}{a}\right)$ to solve for u and find x easily with u since $x = u - \frac{b}{3a}$.

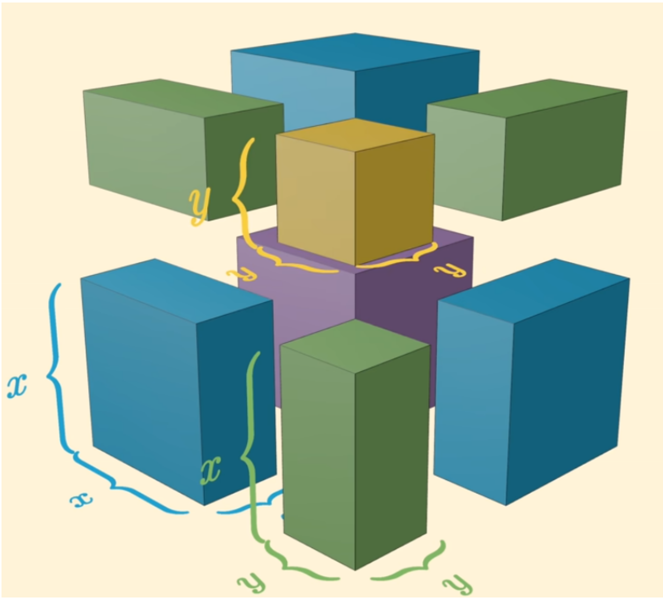

Figure 5: Volumes of the separated parts of the cube

The figure below indicates all the shapes shown above more clearly.

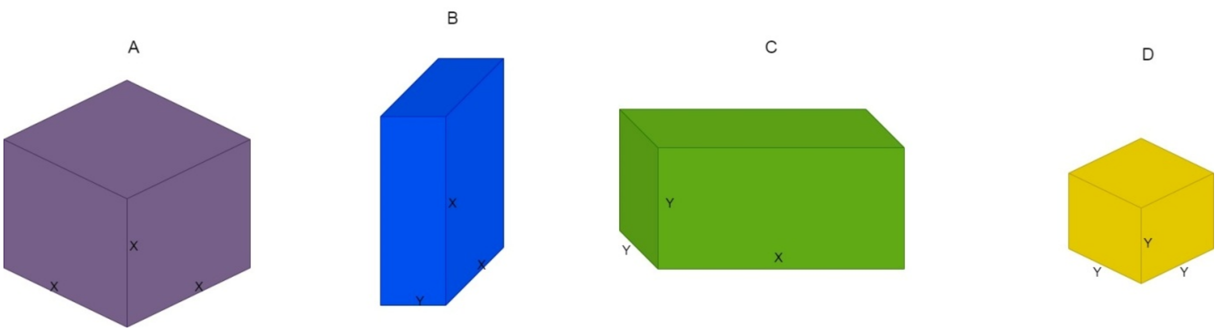

Figure 6: All the different geometric shapes in figure 5, indicating all the sides clearly

In figure 6, A and D are the cubes in figure 5, where A has a volume of x^3 and D has a volume of y^3. The square prism which are B and C are used three times in figure 5, where all the blue square prisms have volumes of $x^2 \times y$ and all the green square prisms have volumes of $x \times y^2$. These 4 figures also indicate the figure 4 which is used for depressed cubic equations.

Let's apply the Cardano's Methodology and other steps to find the solution of a cubic equation:

To solve a third order polynomial, steps (1, 2, 3 and 4) will be used first, then other steps that are used to solve a depressed cubic will be applied. By applying these steps, the cubic polynomial will be transformed into a depressed cubic polynomial.

$$x^3 - 3x^2 + 12x - 36 = 0$$

Normally, after applying step (16), every term will be divided by the leading coefficient, but as the leading coefficient, which is a in methodology, is 1, the division will not be made.

After letting $u - \frac{b}{3a} = x$, which is equal to $u + 1$, the equation will be transformed into following:

$$(u + 1)^3 - 3(u + 1)^2 + 12(u + 1) - 36 = 0$$

According to step (17), it is equivalent to following equation:

$$u^3 + \left(12 - \frac{3^2}{3}\right)u = \left(-\frac{3 \times 12}{3} + \frac{2 \times 3^3}{27} + 36\right)$$

$$u^3 + 9u = 26$$

Which is equal to depressed cubic equation that is observed before. Here, p is equal to 9, and q is equal to 26. After obtaining this equation, solution of depressed cubic will be used again to solve this polynomial. Combining the equation, we will get:

$$(u + y)^3 = y^3 + 26$$

Since,

$$3 \times u \times y \times (u + y) = 9u$$

$$(u + y) = \frac{3}{y}$$

$$9^3 = 27y^6 + 27 \times 26 \times y^3$$

Now, as we have done in depressed cubic, but as the variable u is already in use, let $v = y^3$. Then the following equation will be obtained.

$$729 = 27v^2 + 702v$$

Then,

$$27 = v^2 + 26v$$

$$v = \frac{-26 + \sqrt{26^2 + 4 \times 27}}{2}$$

Which is:

$$v = -13 + \sqrt{169 + 27}$$

After making the addition, the following value of v will be obtained.

$$v = 1$$

Retransforming v to y gives the following:

$$y = \sqrt[3]{1}$$

Which is;

$$y = 1$$

Finally the solution would be found like following:

$$u = \frac{9}{3 \times 1} - 1$$

And it is equal to:

$$u = 2$$

As $u - \frac{b}{3a} = x$, the solution could be found like following:

$$x = 2 + \frac{3}{3 \times 1}$$

$$x = 3$$

Equations That Do Not Satisfy The Methodology

As the methodology, so the beginning of solution of cubic equations uses geometry, there are some situations where the geometry cannot be used to solve the equation. There are two main equation types where the geometric solution cannot be used to solve them. These are equations with negative one or more negative roots, and equations with two complex roots. Having one complex root will already not satisfy the method used in this essay but having one complex root means that the conjugate pair, where only the imaginer part of the number has inverse sign $(+, -)$, of the complex number is also a root for the equation. Applying methodology to these equations may end up with obtaining the true result, however, the geometric part will not be available to use for these equations, which will oppose with the beginning of the solution method, as it is said before. Here are some examples that are opposing with the geometric methodology:

- For quadratic equations, we can use figure 5 to find a situation where the methodology is not satisfied. It could be seen in figure 5 that $(x + 13)^2 = 196$. Geometrically, only $x = 1$ satisfies the equation. However, we can see that $x = -27$ also satisfies the equation. Whether the geometry can be used to find a solution, in some situations, mathematics should be interpreted separately from geometry.

$$x^3 - 6x = 2$$

- For this equation, using digital graphic calculators, such as Desmos and WolframAlpha, it could be seen that all three roots are real numbers, however, using Cardano's Method to solve the equation needs calculations with complex numbers. As solution method is totally based on geometric shapes and complex number had not been found while the solution had been created, the methodology cannot be used for this depressed cubic equation. As stated earlier, further investigations on Cardano's solution method by Bombelli and other mathematicians, these kinds of equations made them to create complex numbers, with putting the geometric proof away.

$$x^3 + 3x^2 + 12x + 36$$

- Here, using Cardano's Method to solve the equation will occur a negative solution, which is $x = -2$, however, using the geometric proof, where the numerical proof came from, we can see that this, obtaining negative geometric areas, was not possible as the geometry had been interpreted with scalar units.

Later Investigations on Cardano's Methodology:

As Cardano found a way to solve some arrangements of cubic polynomials, he realizes that some arrangements could not be easily solved in original method. After transformation of the cubic polynomials to a depressed cubic and some parts of Tartaglia's/Del Ferro's, he obtains some unsolvable terms, which are in form of $a + b\sqrt{-\mathrm{k}}$, where k is a square number. Cardano tries to find a solution for these numbers, but he fails. To find a solution for these terms, he talks about them to Tartaglia. At first, Tartaglia had interpreted that Cardano is not able to use his method right, but after wasting some time on these, he fails to find a reasonable solution, too. Then, Cardano interprets these terms as useless. These negative terms were not included in solutions of some arrangements of cubic equations, and people were not focusing on them, as it is known that they are not affecting the solution, until another mathematician named Bombelli realizes that the sum of these terms which are in form $a + b\sqrt{-\mathrm{k}}$ is equal to zero. Then, he changes these numbers to a form where he separates k and -1 inside of the square root and at the end, he obtains a root in form of $a + m\sqrt{-1}$, which is later transformed to $a + mi$, with the inclusion of other mathematicians and physicists.

<u>Affects of Bombelli's Investigation on Geometric Proof</u>

These negative terms in some arrangements were representing unreal things in real life. In geometrical proof, to complete the cube or the square, negative areas would be needed. For example, in quadratic polynomial, a square with area $30cm^2$ will be used to obtain a square with side lengths 5cm, which leads to usage of area of $-5cm^2$. As these negative areas are not possible in real life, this makes a contradiction to occur in proof of cubic polynomials. The proof and the solution had been derived from the geometry in first place, but after Bombelli's investigation, it shows that geometrical proof is not the real way to show the truth. This led mathematics to develop in different ways. Until this investigation, geometry was the source of truth in mathematics. The geometry had been used to check the mathematical propositions, whether they are true or not, but after this, it is not the direct source of truth anymore.

Conclusion

The investigations, that is used to write this essay, had answered the research question, which is "How does geometry relate to solution of cubic equations?". At the end of the investigations, and research for the essay, it could be said that the objective is successfully achieved.

My objective was to learn the solutions of the cubic equations, and how geometry could be applied to solve these polynomials, and after this essay, I was able to solve some cubic equations and apply geometry with them. I developed my vision about both cubic and quadratic equations as well. Before researching and investigating this topic, I was not much able to solve the polynomials within minutes, without calculating anything on paper, but after the investigations, I try to visualize the polynomial in my mind, and with the usage of geometry, I became capable of solving these polynomials in my mind, with just a few minutes.

Although, the contradiction that is observed after Bombelli's investigation confuses the situation, the method to be useful for most of the cubic equations make this investigation valid to solve cubic equations. To solve those polynomials, which contain the number i, the geometry should not be considered after reaching the roots. The geometry should be used until the part where geometry contradicts with the real life.

According to the study, comprehending the solutions of cubic equations is closely linked to geometry. The geometric interpretation of cubic equations as intersections of curves and the use of a straightedge and compass to construct them have enabled a better understanding of cubic functions and their solutions. Visual representations of cubic equations through graphs and diagrams have also helped to improve our understanding of their geometric properties, including the number and placement of real roots. No information has been omitted in this paraphrased text.

The work has highlighted the significance of geometry in the progress of mathematical knowledge and its historical importance. The foundation for modern geometrical approaches was laid by ancient mathematicians such as Euclid, Archimedes, and Omar Khayyam, who also had an impact on our understanding of cubic equations. The historical context has revealed the challenges and advancements in our knowledge of cubic equations throughout time, indicating the cooperative and iterative nature of mathematical advancement.

In this Extended Essay, the importance of the interdisciplinary nature of mathematics has been emphasized. By combining geometry and algebra, a comprehensive understanding of various mathematical subjects can be achieved. This approach exhibits the versatility and elegance of mathematics, and how different branches of mathematics can be intertwined to offer innovative insights and solutions. It is crucial to recognize the significance of merging different mathematical approaches to gain a deeper understanding of mathematical concepts.

To sum up, the study of the correlation between geometry and solutions of cubic equations has proven that geometry is not just a means of visualization but also an essential element in understanding the behavior and workings of cubic equations and their solutions. The complex interdependence between geometry and mathematics has been unveiled by interpreting cubic equations geometrically, constructing cubic equations using a straightedge and compass, and applying geometric methods to solve cubic equations.

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