International Baccalaureate

Extended Essay

Getting formulas for the convergence of the series defined by multiplications of consecutive terms of an arithmetic sequence in a reciprocal form

Research Question: "Is it possible to obtain the formulas for the convergence of the series defined by multiplications of consecutive terms of an arithmetic sequence in a reciprocal form by using case-by-case proof?

3228 Words

Contents:

Abstract:

This paper presents the two different sequences $\{b_n\}$ and $\{c_n\}$ for which $\{b_n\} = 1 / \prod_{k=n}^{m+n-1} a_k$ and $\{c_n\} = 1 / \prod_{k=1}^{n} a_k$ respectively, where $\{a_n\}$ is a non-zero arithmetic sequence, m is an integer bigger than 1. Furthermore, the aim of this paper is to search for the answer to the research question: "Is it possible to obtain the formulas for the convergence of the series defined by multiplications of consecutive terms of an arithmetic sequence in a reciprocal form by using case-by-case proof?". This investigation relies heavily on series and their convergence to find formulas for the convergence of series $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$. For the convenience to get these formulas, the arithmetic sequence $\{a_n\}$ was used as three different cases. The first case is to choose $\{a_n\} = \{dn\}$ where d is the common difference of the sequence. The second case is to choose $\{a_n\} = \{n + r\}$ where r is a non-negative integer. The third and final case is the combined version of the first and second cases, namely $\{a_n\} = \{dn + r\}$. While investigating a convergence formula for $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$ separately, three different theorems were obtained for each of the three different cases. While discovering the convergence of these series, some mathematical expressions and functions were used. Examples of these are the Maclaurin series, the Complete and Incomplete Gamma function, and the Gamma function's alternate recursive formulas.

Introduction:

This investigation is under the topic "Getting formulas for the convergence of the series defined by multiplications of consecutive terms of an arithmetic sequence in a reciprocal form". Two different but similar series include the consecutive terms of an arithmetic sequence in which each term is multiplied by the other term in a reciprocal format. The reason why I wanted to work on this topic is based on the time when I studied mathematical induction from my book (Haese Mathematics Analysis and Approaches HL) in 2022 to be able to finish most of the math subjects before moving on to that topic as a class. At that time, an example on page 237 of my book caught my attention.

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+...+\frac{1}{n\cdot(n+1)}=\frac{n}{n+1}$$

At the left-hand side of the equation, two numbers are multiplied in the denominator of each term and there is a $n\cdot\left(n+1\right)$ relationship between multiplied terms. The numbers in the denominator of each new term are one more than the numbers in the previous term. This structure can be modeled with a structure with the general term of an arithmetic sequence $a_n = n$.

In this essay, two series will be made in which the consecutive arithmetic sequence terms are multiplied in the denominator. While the number of terms in the denominator is constant in the first form, the multiplied numbers in the denominator will increase by one in each new term. In the second form, one more number will be multiplied by the denominator of each newly added term. In the end, the formula convergence of two different series will be investigated by using case-by-case proof. This method is used because if specific cases are done in order, it is easier to search for a general formula. While investigating the first series, it will mostly be tried to be done through generalization and algebraic operations such as partial fractions and factorization. Both series include algebraic operations such as partial fractions and factorization. However, the methodology for finding the convergence of the 2^{nd} series includes the Maclaurin series and the Gamma function.

The research question of this essay is “Is it possible to obtain the formulas for the convergence of the series defined by multiplications of consecutive terms of an arithmetic sequence in a reciprocal form by using case-by-case proof?”. If the formulas of convergence for these series are obtained, they can be used in different fields of mathematics. For example, these types of series can be helpful for designing new questions in calculus textbooks to enable students to comprehend sequences and series better.

Background Information:

Sequences:

In mathematics, sequences play an essential role in calculus, number theory, and the computer science field. A number sequence is defined as an ordered list of numbers that are represented by a formula. The formula is often named as the general term of a sequence and denoted as a_n. Note that n is defined in a positive integer set. (“Sequences - Sequences in Math Along With Rules, Formulas, and Examples”).

Arithmetic Sequences:

An arithmetic sequence is defined as a sequence in which each term differs from the previous one by the same fixed number. The difference is denoted as d, the common difference (Haese et al.).

Definition: A sequence is an arithmetic $\Leftrightarrow a_{n+1} - a_n = d$ for $n \in \mathbb{Z}^+$

Properties of Arithmetic Sequences:

There are some properties of arithmetic sequences that can be applied to each arithmetic sequence. These properties come in handy when dealing with terms with a high number of terms.

1) General Term of an Arithmetic Sequence:

$$a_n = a_1 + (n-1)d = a_{n-1} + d$$

Let $\{a_n\}$ be an arithmetic sequence. The difference between the second and first terms will have a constant value d. Therefore, the second term can be written in terms of a_1 and d as $a_2 = a_1 + d$. This can be applied to a_3 since the difference between a_3 and a_2 is d:

$$a_3 = a_2 + d$$

Remark that $a_2 = a_1 + d$. So, $a_3 = a_1 + 2d$. If it is generalized, it can be expressed as $a_n = a_1 + (n-1)d$. Additionally, each term can be represented in terms of its previous term since the difference is constant. This type of general term is often referred to as recursive formula: $a_n = a_{n-1} + d$.

2) Midterm of a Sequence (If the Total Number of Terms Is Odd)

In general, $\frac{a_{n-k} + a_{n+k}}{2} = a_n$

Proof:

Let k be an even number.

$$
\begin{array}{l}
\quad a_{n-k} = a_1 + (n-k-1)d \\
\underline{+a_{n+k} = a_1 + (n+k-1)d} \\
a_{n-k} + a_{n+k} = 2a_1 + 2(n-1)d \\
\frac{a_{n-k} + a_{n+k}}{2} = a_1 + (n-1)d = a_n
\end{array}
$$

Series:

A series is defined as the sum of the terms of a sequence. Moreover, the series can be considered as a sequence of partial sums of terms of the sequence. In general, for a finite sequence with n terms, the series will be represented as $a_1 + a_2 + a_3 + a_4 + ... + a_n$. Additionally, for an infinite sequence, the series will be $a_1 + a_2 + a_3 + a_4 + ...$. Series are usually denoted as $\{S_n\}$. The convergence of a series is defined as the $\lim_{n\to\infty} \{S_n\}$. The infinite series can be calculated on some occasions, and sometimes not. If it can be calculated, it is called convergent series. If not, then it is called divergent series. For representing a series, there is a notation called sigma notation, which is used for ruled summation. It is represented by the Greek letter sigma, Σ (Adams and Essex).

$$\sum_{i=m}^{n} f(i) = f(m) + f(m+1) + f(m+2) + f(m+3) + ... + f(n)$$

The variable i is called the index of summation. This is used for replacing the integers $m, m+1, m+2, ...n$ successively to sum the results. The constant m is called the lower limit and n is called the upper limit (Adams and Essex).

Properties of Sigma Notation:

$$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

For a constant c, $\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$ and $\sum_{k=1}^{n} c = cn$

There are many types of series. However, in this essay, only four series will be explained. These are the arithmetic series, the geometric series, the power series, and the Taylor/Maclaurin series.

Arithmetic Series:

According to the definition of series, arithmetic series is the sum of terms of arithmetic sequences (Haese et al.). The partial sum of arithmetic sequence terms can be denoted by:

$\sum_{i=m}^{n} a_m + a_{m+1} + a_{m+2} + ... + a_n$ where $a_n = a_1 + (n-1)d$

Finite Arithmetic Series of a Given Arithmetic Sequence:

Let the arithmetic sequence has the first term a_1 and common difference d. Since each term of the sequence can be written in terms of the first term and common difference, $\{S_n\}$ the sequence of partial sums of the series $\sum_{n=1}^{\infty} a_n$ can be represented as:

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + ... + (a_n - 2d) + (a_n - d) + a_n \; (1)$$

S_n can be rewritten as $a_n + (a_n - d) + (a_n - 2d) + ... + (a_1 + 2d) + (a_1 + d) + a_1$ (2) {By reversing the terms}

If equation (1) and (2) is added vertically, the final form of the equation will be:

$$2S_n = (a_1 + a_n) + (a_1 + a_n) + ... + (a_1 + a_n) + (a_1 + a_n)$$

There are n terms on the right-hand side of the equation. After the addition of n terms is written as $n(a_1 + a_n)$, divide both sides by two:

$$2S_n = n(a_1 + a_n)$$
$$S_n = \frac{n(a_1 + a_n)}{2}$$

Additionally, $a_n = a_1 + (n-1)d$. Hence, a_n can be written both as $S_n = \frac{n(a_1 + a_n)}{2}$ or $S_n = \frac{n(2a_1 + (n-1)d)}{2}$.

Geometric Series:

Geometric series are the series in the form of $\sum_{k=1}^{n} ar^{n-1} = a + ar + ar^2 + ... + ar^{n-1}$ and its n[th] term is $a_n = ar^{n-1}$. The coefficient a is called the first term and the number r is called the common ratio. If the variable n approaches infinity, then it is called infinite geometric series. The series converges to a finite number in the form of $\frac{a}{1-r}$ if $|r| < 1$. Otherwise, it will diverge (Adams and Essex).

Power Series:

Power series are the series in form of:

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + ...$$

The number c is often called the center of convergence of the series. Terms of the series are functions of x. Therefore, it may converge or not converge depending on each value of x. For example, if $|r| < 1$, then:

$$1 + x + x^2 + x^3 + ... = \frac{1}{1-x}$$

The left-hand side of the equation contains a geometric series which is represented as a power series as a representation of $\frac{1}{1-x}$ in powers of x for $|x| < 1$ (Adams and Essex).

Taylor/Maclaurin Series:

Some functions can be represented as a power series and they converge under an interval. This will enable to approximate certain values of a function that are not easy to be calculated. Because of this, Taylor/Maclaurin series will be crucial for approximating a function. First of all, let the function *f* has a power series representation at the center $x = c$ and has derivatives of every order.

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \dots$$

If $x = c$, the function $f(c)$ would be equal to the coefficient a_0.

If the equation is differentiated, it can be seen that

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} a_n(x-c)^n\right) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \dots$$

The derivative at $x = c$ equals to a_1. Therefore, if the derivative of the series is $f'(c)$.

If the series is differentiated again and again, a factorial relationship after each derivative can be seen on the terms:

$$\frac{d^2}{dx^2}\left(\sum_{n=1}^{\infty} a_n(x-c)^n\right) = 2a_2 + 3\cdot 2a_3(x-c) + 4\cdot 3a_4(x-c)^2 + \dots$$

$$\frac{d^3}{dx^3}\left(\sum_{n=2}^{\infty} a_n(x-c)^n\right) = 3\cdot 2a_3 + 4\cdot 3\cdot 2a_4(x-c) + 5\cdot 4\cdot 3a_5(x-c)^2 + \dots$$

At $x = c$, derivatives will be:

$$\frac{d^2}{dx^2}\left(\sum_{n=1}^{\infty} a_n(x-c)^n\right) = 2a_2 + 3\cdot 2a_3(c-c) + 4\cdot 3a_4(c-c)^2 + ... = 2a_2$$

$$\frac{d^3}{dx^3}\left(\sum_{n=2}^{\infty} a_n(x-c)^n\right) = 3\cdot 2a_3 + 4\cdot 3\cdot 2a_4(c-c) + 5\cdot 4\cdot 3a_5(c-c)^2 + ... = 3\cdot 2a_3$$

Derivatives will be equal to $f''(c)$ and $f'''(c)$ consecutively. Additionally,

$a_2 = \frac{f''(c)}{2}$ and $a_3 = \frac{f'''(c)}{3\cdot 2}$. The factorial relationship of each differentiation to the

coefficients can be generalized as $a_n = \frac{f^{(n)}(c)}{n!}$.

The overall series can be written as:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!} + ...$$

This series is called the Taylor series. If the center of the series is equal to zero, then the series is called the Maclaurin series. To have a better understanding of these series, an example that will be used in the investigation is given (Strang and Herman).

In the investigation, the function $f(x) = e^x$ will be used in a Maclaurin series form. In order to do this, the generalized pattern of higher derivatives of e^x must be found.

The first derivative of e^x is equal to e^x. Moreover, the second and all higher derivatives of e^x is equal to themselves. If c is assumed to be zero (due to the definition of the Maclaurin series),

$$\begin{aligned} f'(0) &= e^0 = 1 \\ f''(0) &= e^0 = 1 \\ f'''(0) &= e^0 = 1 \\ &\dots \\ f^{(n)}(0) &= e^0 = 1 \end{aligned}$$

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n = f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}(x)^3 + \dots = e^x \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = e^x \end{aligned}$$

To visualize the Maclaurin series, a graphic display calculator program will be used to graph e^x and its Maclaurin series representation.

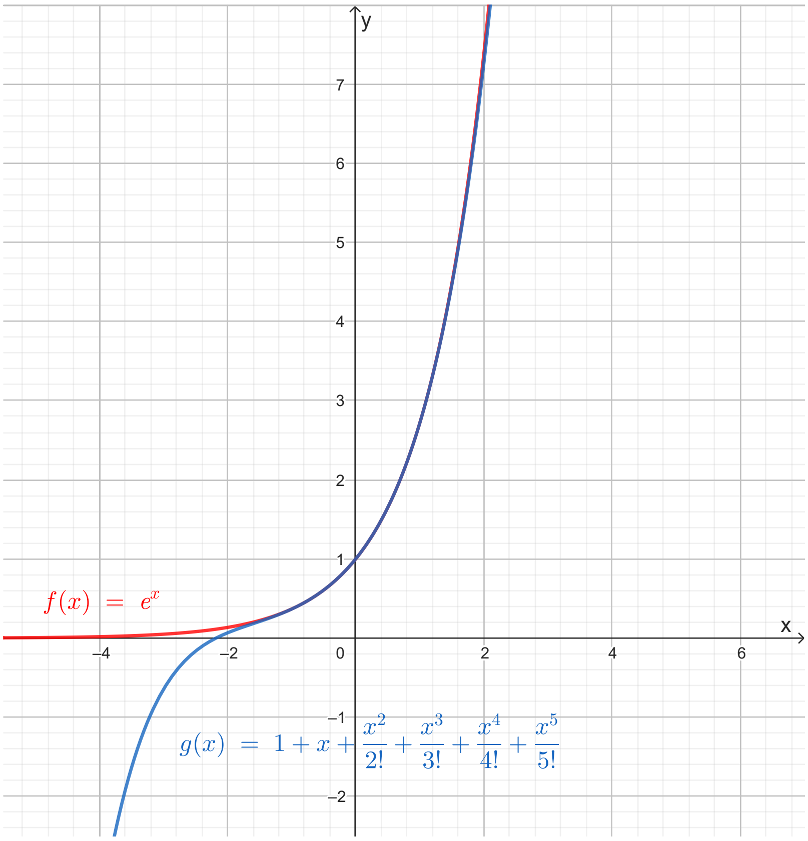

Figure 1: A screenshot of GDC program where the functions were plotted

The function and its Maclaurin series representation (up to $n = 6$) are sketched. The original function and series representation is seen to be intersecting at an interval. However, note that there is a marginal difference between them which is not noticeable until it is zoomed in. If the number n is increased, the difference between the original and the representation will decrease. Remark that writing the series up to a finite number will only approximate the original function. However, increasing the terms will make a better approximation of the original function.

Investigation:

1) **The convergence of** $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{\prod_{k=n}^{m+n-1} a_k}$

To get a general formula, the convergence of the series $\sum_{n=1}^{\infty} b_n$ will be investigated case by case. First, the arithmetic sequences $\{a_n\} = \{d \cdot n\}$ will be used where d is the common difference. Then, the arithmetic sequence $\{a_n\} = \{n + r\}$ where r is a non-negative integer will be looked at. And finally, the general arithmetic sequence $\{a_n\} = \{d \cdot n + r\}$ will be used for any non-zero real d.

Case 1:

Let $\{a_n\} = \{d \cdot n\}$. Before getting a general solution, the case for some specific values of d will be investigated. Now, suppose that $d = 1$. Since $\{a_n\} = \{n\}$,

$b_n = \frac{1}{\prod_{k=n}^{m+n-1} k} = \frac{1}{n(n+1)\cdots(n+m-1)}$. So, $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n(n+1)\cdots(n+m-1)}$.

To find the convergence of the series, some algebraic operations on b_n needs to be done. Note that,

$$\begin{aligned}\frac{1}{n(n+1)\cdots(n+m-1)} &= \frac{1}{(n+1)\cdots(n+m-2)}\frac{1}{n(n+m-1)} \\ &= \frac{1}{(n+1)\cdots(n+m-2)}\frac{1}{m-1}\left(\frac{1}{n}-\frac{1}{n+m-1}\right) \\ &= \frac{1}{m-1}\left[\frac{1}{n(n+1)\cdots(n+m-2)}-\frac{1}{(n+1)(n+2)\cdots(n+m-1)}\right]\end{aligned}$$

So, the partial sum of the series becomes:

$$\begin{aligned}\sum_{n=1}^{N} b_n &= \frac{1}{m-1}\left[\frac{1}{n(n+1)\cdots(n+m-2)}-\frac{1}{(n+1)(n+2)\cdots(n+m-1)}\right] \\ &= \frac{1}{m-1}\left[\sum_{n=1}^{N}\frac{1}{n(n+1)\cdots(n+m-2)}-\sum_{n=1}^{N}\frac{1}{(n+1)(n+2)\cdots(n+m-1)}\right] \\ &= \frac{1}{m-1}\left[\frac{1}{(m-1)!}-\frac{1}{(N+1)\cdots(N+m-1)}\right]\end{aligned}$$

Since $\sum_{n=1}^{\infty} b_n = \lim_{N\to\infty}\sum_{n=1}^{N} b_n$,

$$\begin{aligned}\sum_{n=1}^{\infty} b_n &= \lim_{N\to\infty}\frac{1}{m-1}\left[\frac{1}{(m-1)!}-\frac{1}{(N+1)\cdots(N+m-1)}\right] \\ &= \frac{1}{(m-1)(m-1)!}\end{aligned}$$

Thus, the following theorem will be:

Theorem 1:

The series $\sum_{n=1}^{\infty}\frac{1}{n(n+1)\cdots(n+m-1)}$ converges to $\frac{1}{(m-1)(m-1)!}$ for all $m > 1$.

To illustrate, the series will be demonstrated for the values 2, 3, 4, and 5 for m :

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots=\frac{1}{1\cdot 1!}=1$$

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)}=\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\cdots=\frac{1}{2\cdot 2!}=\frac{1}{4}$$

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)(n+3)}=\frac{1}{1\cdot 2\cdot 3\cdot 4}+\frac{1}{2\cdot 3\cdot 4\cdot 5}+\cdots=\frac{1}{3\cdot 3!}=\frac{1}{18}$$

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)\cdots(n+4)}=\frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6}+\cdots=\frac{1}{4\cdot 4!}=\frac{1}{96}.$$

Now, suppose that $d = 2$. Since $\{a_n\} = \{2n\}$, the sequence will be:

$$b_n = \frac{1}{\prod_{k=n}^{m+n-1} 2k} = \frac{1}{2^m}\cdot\frac{1}{n(n+1)\cdots(n+m-1)}. \text{ So, } \sum_{n=1}^{\infty} b_n = \frac{1}{2^m}\sum_{n=1}^{\infty}\frac{1}{n(n+1)\cdots(n+m-1)}.$$

Since theorem 1 says that $\sum_{n=1}^{\infty}\frac{1}{n(n+1)\cdots(n+m-1)}$ converges to $\frac{1}{(m-1)(m-1)!}$ for all $m > 1$, for $d = 2$, the series converges to $\frac{1}{2^m(m-1)(m-1)!}$.

For example, if $m = 3$ is chosen, then:

$$\sum_{n=1}^{\infty} \frac{1}{2n \cdot 2(n+1) \cdot 2(n+2)} = \frac{1}{2 \cdot 4 \cdot 6} + \frac{1}{4 \cdot 6 \cdot 8} + \cdots = \frac{1}{2^3 \cdot 2 \cdot 2!} = \frac{1}{32}.$$

The value of $d = 2$ shows that for an arbitrary non-zero d, one can write

$b_n = \frac{1}{\prod_{k=n}^{m+n-1} d \cdot k} = \frac{1}{d^m} \cdot \frac{1}{n(n+1)\cdots(n+m-1)}$ which implies that the series $\sum_{n=1}^{\infty} b_n$

converges to $\frac{1}{d^m (m-1)(m-1)!}$. Thus, case 1 is done and the following theorem can be

written as:

<u>Theorem 2:</u>

The series $\sum_{n=1}^{\infty} \frac{1}{dn \cdot d(n+1) \cdots d(n+m-1)}$ converges to $\frac{1}{d^m (m-1)(m-1)!}$ for all $m >$

1 and non-zero d.

<u>Case 2:</u>

Let $\{a_n\} = \{n + r\}$ where r is a non-negative integer.

Then $b_n = \frac{1}{\prod_{k=n}^{m+n-1} (k+r)} = \frac{1}{(n+r)(n+r+1)\cdots(n+r+m-1)}$. So,

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{(n+r)(n+r+1)\cdots(n+r+m-1)}.$$

Note that,

$$\frac{1}{(n+r)(n+r+1)\cdots(n+r+m-1)} = \frac{1}{(n+r+1)\cdots(n+r+m-2)}\frac{1}{(n+r)(n+r+m-1)}$$
$$= \frac{1}{(n+r+1)\cdots(n+r+m-2)}\frac{1}{m-1}\left(\frac{1}{n+r} - \frac{1}{n+r+m-1}\right)$$
$$= \frac{1}{m-1}\left[\frac{1}{(n+r)\cdots(n+r+m-2)} - \frac{1}{(n+r+1)\cdots(n+r+m-1)}\right]$$

So, the partial sum of the series becomes:

$$\sum_{n=1}^{N} b_n = \frac{1}{m-1}\left[\frac{1}{(n+r)\cdots(n+r+m-2)} - \frac{1}{(n+r+1)\cdots(n+r+m-1)}\right]$$
$$= \frac{1}{m-1}\left[\sum_{n=1}^{N}\frac{1}{(n+r)\cdots(n+r+m-2)} - \sum_{n=1}^{N}\frac{1}{(n+r+1)\cdots(n+r+m-1)}\right]$$
$$= \frac{1}{m-1}\left[\frac{r!}{(r+m-1)!} - \frac{1}{(N+r+1)\cdots(N+r+m-1)}\right]$$

Since $\sum_{n=1}^{\infty} b_n = \lim_{N\to\infty}\sum_{n=1}^{N} b_n$, the equation will be:

$$\sum_{n=1}^{\infty} b_n = \lim_{N\to\infty}\frac{1}{m-1}\left[\frac{r!}{(r+m-1)!} - \frac{1}{(N+r+1)\cdots(N+r+m-1)}\right]$$
$$= \frac{r!}{(m-1)(r+m-1)!}$$

Thus, case 2 is done and the following theorem will be:

<u>Theorem 3:</u>

The series $\sum_{n=1}^{\infty}\frac{1}{(n+r)\cdots(n+r+m-1)}$ converges to $\frac{r!}{(m-1)(r+m-1)!}$ for all $m >$ 1, where r is a non-negative integer.

To illustrate,

for $m = 3$ and $r = 4$, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+4)(n+5)(n+6)} = \frac{1}{5\cdot 6\cdot 7} + \frac{1}{6\cdot 7\cdot 8} + \cdots = \frac{4!}{2\cdot 6!} = \frac{1}{60}$

for $m = 4$ and $r = 7$, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+7)\cdots(n+10)} = \frac{1}{8\cdots 11} + \frac{1}{9\cdots 12} + \cdots = \frac{7!}{3\cdot 10!} = \frac{1}{2160}$.

<u>Case 3:</u>

Let $\{a_n\} = \{dn + r\}$ is a non-zero sequence and d is a non-zero real.

Then $b_n = \dfrac{1}{\prod_{k=n}^{m+n-1} (dk + r)} = \dfrac{1}{(dn+r)(d(n+1)+r)\cdots(d(n+m-1)+r)}$. So,

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{(dn+r)(d(n+1)+r)\cdots(d(n+m-1)+r)}.$$

Note that,

$$\begin{aligned}
\frac{1}{(dn+r)\cdots(d(n+m-1)+r)} &= \frac{1}{(d(n+1)+r)\cdots(d(n+m-2)+r)} \frac{1}{(dn+r)(d(n+m-1)+r)} \\
&= \frac{1}{(d(n+1)+r)\cdots(d(n+m-2)+r)} \frac{1}{d(m-1)} \left(\frac{1}{dn+r} - \frac{1}{d(n+m-1)+r} \right) \\
&= \frac{1}{d(m-1)} \left[\frac{1}{(dn+r)\cdots(d(n+m-2)+r)} - \frac{1}{(d(n+1)+r)\cdots(d(n+m-1)+r)} \right]
\end{aligned}$$

So, the partial sum of the series becomes:

$$\sum_{n=1}^{N} b_n = \frac{1}{d(m-1)}\left[\frac{1}{(dn+r)\cdots(d(n+m-2)+r)} - \frac{1}{(d(n+1)+r)\cdots(d(n+m-1)+r)}\right]$$
$$= \frac{1}{d(m-1)}\left[\sum_{n=1}^{N}\frac{1}{(dn+r)\cdots(d(n+m-2)+r)} - \sum_{n=1}^{N}\frac{1}{(d(n+1)+r)\cdots(d(n+m-1)+r)}\right]$$
$$= \frac{1}{d(m-1)}\left[\frac{1}{(d+r)\cdots(d(m-1)+r)} - \frac{1}{(d(N+1)+r)\cdots(d(N+m-1)+r)}\right]$$

Since $\sum_{n=1}^{\infty} b_n = \lim_{N\to\infty}\sum_{n=1}^{N} b_n$, the series will be:

$$\sum_{n=1}^{\infty} b_n = \lim_{N\to\infty}\frac{1}{d(m-1)}\left[\frac{1}{(d+r)\cdots(d(m-1)+r)} - \frac{1}{(d(N+1)+r)\cdots(d(N+m-1)+r)}\right]$$
$$= \frac{1}{d(m-1)}\frac{1}{(d+r)(2d+r)\cdots(d(m-1)+r)}$$

Thus, case 3 is done and the following final theorem for $\sum_{n=1}^{\infty} b_n$ is finally found.

<u>Theorem 4:</u>

The series $\sum_{n=1}^{\infty}\frac{1}{(dn+r)(d(n+1)+r)\cdots(d(n+m-1)+r)}$ converges to

$\frac{1}{d(m-1)}\frac{1}{(d+r)(2d+r)\cdots(d(m-1)+r)}$ for all $m>1$, where $\{a_n\} = \{dn+r\}$ is a non-zero

sequence and d is a non-zero real.

<u>Examples:</u>

a) Evaluate $\sum_{n=1}^{\infty}\frac{1}{(2n+3)(2n+5)(2n+7)(2n+9)}$.

Solution: Since $a_n = 2n+3$, the values for d, r, and m will be $d = 2$, $r = 3$, and $m = 4$. Thus,

$$\sum_{n=1}^{\infty} \frac{1}{(2n+3)(2n+5)(2n+7)(2n+9)} = \frac{1}{5\cdot 7\cdot 9\cdot 11} + \frac{1}{7\cdot 9\cdot 11\cdot 13} + \cdots$$
$$= \frac{1}{2(4-1)}\frac{1}{5\cdot 7\cdot 9}$$
$$= \frac{1}{1890}$$

b) Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(-3n+2)(-3n-1)(-3n-4)}$.

Solution: Since $a_n = -3n+2$, the values for d, r, and m will be $d = -3$, $r = 2$, and $m = 3$. Thus,

$$\sum_{n=1}^{\infty} \frac{1}{(-3n+2)(-3n-1)(-3n-4)} = \frac{1}{(-1)\cdot(-4)\cdot(-7)} + \frac{1}{(-7)\cdot(-10)\cdot(-13)} + \cdots$$
$$= \frac{1}{-3(3-1)}\frac{1}{(-1)\cdot(-4)}$$
$$= -\frac{1}{24}$$

c) Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\left(4n-\frac{2}{3}\right)\left(4n+\frac{10}{3}\right)\left(4n+\frac{22}{3}\right)\left(4n+\frac{34}{3}\right)\left(4n+\frac{46}{3}\right)}$.

Solution: Since $a_n = 4n - \frac{2}{3}$, the values for d, r, and m will be $d = 4$, $r = -\frac{2}{3}$ and $m = 5$. Thus,

$$\sum_{n=1}^{\infty}\frac{1}{\left(4n-\frac{2}{3}\right)\left(4n+\frac{10}{3}\right)\left(4n+\frac{22}{3}\right)\left(4n+\frac{34}{3}\right)\left(4n+\frac{46}{3}\right)}$$

$$=\frac{1}{\frac{10}{3}\cdot\frac{22}{3}\cdot\frac{34}{3}\cdot\frac{46}{3}\cdot\frac{58}{3}}+\frac{1}{\frac{22}{3}\cdot\frac{34}{3}\cdot\frac{46}{3}\cdot\frac{58}{3}\cdot\frac{70}{3}}+\cdots$$

$$=\frac{1}{4(5-1)}\frac{1}{\frac{10}{3}\cdot\frac{22}{3}\cdot\frac{34}{3}\cdot\frac{46}{3}}$$

$$=\frac{81}{5505280}$$

2) The convergence of $\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} \frac{1}{\prod_{k=1}^{n} a_k}$

To get a general formula, the convergence of the series $\sum_{n=1}^{\infty} c_n$ will be investigated through case-by-case proof. First, the arithmetic sequences $\{a_n\} = \{d \cdot n\}$ will be used where d is the common difference. Then, the arithmetic sequence $\{a_n\} = \{n + r\}$ where r is a non-negative integer will be looked at. And finally, the general arithmetic sequence $\{a_n\} = \{d \cdot n + r\}$ will be used for any non-zero real d.

Case 1:

Let $\{a_n\} = \{d \cdot n\}$. Before getting a general solution, the case for some specific values of d will be investigated. Now, suppose that $d = 1$. Since $\{a_n\} = \{n\}$, c_n will be

$$c_n = \frac{1}{\prod_{k=1}^{n} k} = \frac{1}{n!}.$$

So,

$$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

Since the Maclaurin series representation of e^x is $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \ldots$,

e can be written as $e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$. Therefore, the sequence $\sum_{n=1}^{\infty} c_n$

converges to $e - 1$. In other words,

$$\begin{aligned} \sum_{n=1}^{\infty} c_n &= \sum_{n=1}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots \\ &= -1 + \sum_{n=0}^{\infty} \frac{1}{n!} \\ &= -1 + e \end{aligned} .$$

Now suppose that $d = 2$. Then $c_n = \frac{1}{\prod_{k=1}^{n} 2k} = \frac{1}{2^n n!} = \frac{(1/2)^n}{n!}$. So, following equation

will be:

$$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} \frac{(1/2)^n}{n!} = \frac{1}{2} + \frac{(1/2)^2}{2!} + \frac{(1/2)^3}{3!} + \cdots$$

Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ for all $x \in \mathbb{R}$ and by taking $x = \frac{1}{2}$, it is

obtained that:

$$e^{1/2} = \sum_{n=0}^{\infty} \frac{(1/2)^n}{n!} = 1 + \frac{1}{2} + \frac{(1/2)^2}{2!} + \frac{(1/2)^3}{3!} + \cdots$$

Thus,

$$\begin{aligned}\sum_{n=1}^{\infty} c_n &= \sum_{n=1}^{\infty} \frac{(1/2)^n}{n!} = \frac{1}{2} + \frac{(1/2)^2}{2!} + \frac{(1/2)^3}{3!} + \cdots \\ &= -1 + \sum_{n=0}^{\infty} \frac{(1/2)^n}{n!} \\ &= -1 + e^{1/2}\end{aligned}.$$

In general, for an arbitrary non-zero d, $c_n = \frac{1}{\prod_{k=1}^{n} dk} = \frac{1}{d^n n!} = \frac{(1/d)^n}{n!}$. Thus, $\sum_{n=1}^{\infty} c_n$ will be:

$$\begin{aligned}\sum_{n=1}^{\infty} c_n &= \sum_{n=1}^{\infty} \frac{(1/d)^n}{n!} = \frac{1}{d} + \frac{(1/d)^2}{2!} + \frac{(1/d)^3}{3!} + \cdots \\ &= -1 + \sum_{n=0}^{\infty} \frac{(1/d)^n}{n!} \\ &= -1 + \sqrt[d]{e}\end{aligned}$$

Thus, case 1 is done and the following theorem is obtained:

<u>Theorem 5:</u>

The series $\sum_{n=1}^{\infty} \frac{1}{d^n n!}$ converges to $\sqrt[d]{e} - 1$ for all non-zero d.

<u>Examples:</u>

a) $\frac{1}{3} + \frac{1}{3 \cdot 6} + \frac{1}{3 \cdot 6 \cdot 9} + \cdots \quad = \sum_{n=1}^{\infty} \frac{1}{3^n n!}$

$$= \sqrt[3]{e} - 1 \approx 0.39561$$

b) $\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}\cdot 2\sqrt{2}}+\frac{1}{\sqrt{2}\cdot 2\sqrt{2}\cdot 3\sqrt{2}}+\cdots \;=\sum_{n=1}^{\infty}\frac{1}{\sqrt{2}^{n}\,n!}$

$$=e^{1/\sqrt{2}}-1\approx 1.02811$$

c) $\frac{1}{-5}+\frac{1}{(-5)\cdot(-10)}+\frac{1}{(-5)\cdot(-10)\cdot(-15)}+\cdots \;=\sum_{n=1}^{\infty}\frac{1}{(-5)^{n}n!}$

$$=e^{-1/5}-1\approx -0.18127$$

Case 2:

Let $\{a_n\}=\{n+r\}$ where r is a non-negative integer. Since $\{a_n\}=\{n+r\}$, the sequence will be:

$$c_n=\frac{1}{\prod_{k=1}^{n}(k+r)}=\frac{r!}{(n+r)!}.$$

So,

$$\sum_{n=1}^{\infty}c_n=\sum_{n=1}^{\infty}\frac{r!}{(n+r)!}=r!\sum_{n=1}^{\infty}\frac{1}{(n+r)!}=r!\left(\frac{1}{(1+r)!}+\frac{1}{(2+r)!}+\cdots\right).$$

Note that,

$$\frac{1}{(1+r)!}+\frac{1}{(2+r)!}+\cdots=\sum_{k=0}^{\infty}\frac{1}{k!}-\sum_{k=0}^{r}\frac{1}{k!}=e-\sum_{k=0}^{r}\frac{1}{k!}.$$

Hence, $\sum_{n=1}^{\infty}c_n=\sum_{n=1}^{\infty}\frac{r!}{(n+r)!}$ converges to $r!\left(e-\sum_{k=0}^{r}\frac{1}{k!}\right)$ for all non-zero integers r.

Thus, case 2 is done and the following theorem will be:

<u>Theorem 6:</u>

The series $\displaystyle\sum_{n=1}^{\infty}\frac{r!}{(n+r)!}$ converges to $\displaystyle r!\left(e-\sum_{k=0}^{r}\frac{1}{k!}\right)$ for all non-zero integers r.

<u>Examples:</u>

a) $$\frac{1}{5}+\frac{1}{5\cdot 6}+\frac{1}{5\cdot 6\cdot 7}+\frac{1}{5\cdot 6\cdot 7\cdot 8}+\cdots=\sum_{n=1}^{\infty}\frac{4!}{(n+4)!}$$

$$=4!\left(e-\sum_{n=0}^{4}\frac{1}{n!}\right)$$

$$=24e-65\approx 0.23876$$

b) $$\frac{1}{8}+\frac{1}{8\cdot 9}+\frac{1}{8\cdot 9\cdot 10}+\frac{1}{8\cdot 9\cdot 10\cdot 11}\cdots \quad =\sum_{n=1}^{\infty}\frac{7!}{(n+7)!}$$

$$=7!\left(e-\sum_{n=0}^{7}\frac{1}{n!}\right)$$

$$=5040e-13700\approx 0.14042$$

<u>Case 3:</u>

Let $\{a_n\}=\{dn+r\}$ is a non-zero sequence and d is a non-zero real.

Then $\displaystyle c_n=\frac{1}{\prod_{k=1}^{n}(dk+r)}=\frac{1}{(d+r)(2d+r)\cdots(dn+r)}$. So,

$$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} \frac{1}{(d+r)(2d+r)\cdots(dn+r)}.$$

Before the convergence of the series, factorials must be extended to real numbers. This was done by the famous function named Gamma. In fact, the Gamma function extends factorials to complex numbers, but for convenience, the domain was restricted to real numbers only.

<u>Definition 1 (Gamma Function):</u>

Let x be a positive real number. Then the function Γ defined by:

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$$

is called the Gamma function (Orloff).

Note that two important results for the Gamma function have been gotten:

1. $\Gamma(1) = \int_0^{\infty} e^{-t} dt = \lim_{x\to\infty}\left(-\frac{1}{e^t}\Big|_0^x\right) = \lim_{x\to\infty}\left(-\frac{1}{e^x}+1\right) = 1$

2. $\Gamma(x+1) = \int_0^{\infty} t^x e^{-t} dt$

$$
\begin{aligned}
&= \lim_{b\to\infty}\left(-\frac{t^x}{e^t}\Bigg|_0^b\right) + \int_0^\infty x t^{x-1}e^{-t}dt \\
&= \lim_{b\to\infty}\left(-\frac{b^x}{e^b}\right) + x\int_0^\infty t^{x-1}e^{-t}dt \\
&= 0 + x\Gamma(x) \\
&= x\Gamma(x)
\end{aligned}
$$

By combining these two results, one can see that if n is a positive integer, then $\Gamma(n) = (n-1)!$. For example,

$$\Gamma(2) = 1\cdot\Gamma(1) = 1$$

$$\Gamma(3) = 2\cdot\Gamma(2) = 2\cdot 1 = 2!$$

$$\Gamma(4) = 3\cdot\Gamma(3) = 3\cdot 2! = 3!$$

On the other hand, definition 1 does not allow to define gamma function for non-positive reals. In order to explain why, note that:

if $x = 0$, then $\Gamma(0) = \int_0^\infty t^{-1}e^{-t}dt = \lim_{b\to\infty}\left(-\frac{1}{te^t}\Bigg|_0^b\right) - \int_0^\infty t^{-2}e^{-t}dt$. But $\lim_{b\to\infty}\left(-\frac{1}{te^t}\Bigg|_0^b\right)$ does not exist. And if $x < 0$, then $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt = \lim_{b\to\infty}\left(-\frac{t^x}{xe^t}\Bigg|_0^b\right) + \frac{1}{x}\int_0^\infty t^x e^{-t}dt$. But $\lim_{b\to\infty}\left(-\frac{t^x}{xe^t}\Bigg|_0^b\right)$ does not exist.

However, the definition of the Gamma function to negative reals can be extended by the following definition.

Definition 2:

Let x be a real number except zero or a negative integer. Then,

$$\Gamma(x) = \frac{1}{x}\Gamma(x+1)$$

Note that $\Gamma(0)$, $\Gamma(-1)$, $\Gamma(-2),\cdots$ are still undefined. However, for example $\Gamma\left(-0.5\right)$, can be calculated. Before doing this, a useful recursion formula needs to be mentioned, which is named Euler's reflection formula.

Euler's reflection formula:

$\Gamma(1-x)\Gamma(x) = \dfrac{\pi}{\sin \pi x}$ for $x \notin \mathbb{Z}$ (Gamma Function | Brilliant Math and Science Wiki)

For example, $\Gamma\left(1-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \dfrac{\pi}{\sin\frac{\pi}{2}} \Rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. So, by the definition 2:

$$\Gamma\left(-\frac{1}{2}\right) = \frac{1}{-\frac{1}{2}}\Gamma\left(-\frac{1}{2}+1\right) = -2\Gamma\left(\frac{1}{2}\right) = -2\sqrt{\pi}\ .$$

In order to find the formula for case three, two new functions derived from the Gamma function must be introduced. They are called Upper and Lower Incomplete Gamma Functions. They are derived by the separation of the Gamma function, a definite integral.

Definition 3 (Upper and Lower Incomplete Gamma Functions)

Let x and k be positive real numbers.

Then the upper incomplete gamma function is defined as:

$$\Gamma(x,k)=\int_k^\infty t^{x-1}e^{-t}dt$$

The lower incomplete gamma function is defined as:

$$\gamma(x,k)=\int_0^k t^{x-1}e^{-t}dt$$

It is observed that $\Gamma(x)=\Gamma(x,k)+\gamma(x,k)$. The property of separation of definite integrals comes in handy with this observation. On the other hand, the following recursion formula for the incomplete lower gamma function is needed for the convergence of the series $\sum_{n=1}^{\infty} c_n$.

Theorem 8:

$$\gamma(x+1,k)=x\gamma(x,k)-k^x e^{-k}$$

Proof:

By definition, $\gamma(x+1,k)=\int\limits_0^k t^x e^{-t}dt$. If integration by part is used for $u=t^x$ and $e^{-t}dt=dv$,

$$\gamma(x+1,k)=\int\limits_0^k t^x e^{-t}dt=\left(-t^x e^{-t}\right)_0^k+x\int\limits_0^k t^{x-1}e^{-t}dt$$

which follows the recursion result $\gamma(x+1,k)=x\gamma(x,k)-k^x e^{-k}$.

The last instrument to deduce a formula for the series $\sum_{n=1}^{\infty}c_n$ is the following theorem.

Theorem 9:

$$\gamma(x,k)=k^x e^{-k}\sum_{i=0}^{\infty}\frac{k^i}{x(x+1)\cdots(x+i)}$$

Proof:

If the recursion formula given by Theorem 8 for $\gamma(x+2,k)$ is used, the following equation will be gotten:

$$\begin{aligned}\gamma(x+2,k)&=(x+1)\gamma(x+1,k)-k^{x+1}e^{-k}\\&=(x+1)x\gamma(x,k)-k^x e^{-k}(k+x+1)\\&=(x+1)x\left[\gamma(x,k)-k^x e^{-k}\left(\frac{1}{x}+\frac{k}{x(x+1)}\right)\right]\end{aligned}$$

So, $\gamma(x,k) = \dfrac{\gamma(x+2,k)}{(x+1)x} + k^x e^{-k}\left(\dfrac{1}{x} + \dfrac{k}{x(x+1)}\right)$

Similarly, for $\gamma(x+3,k)$, the following equation will be gotten:

$$\begin{aligned}\gamma(x+3,k) &= (x+2)\gamma(x+2,k) - k^{x+2}e^{-k} \\ &= (x+2)(x+1)x\gamma(x,k) - k^x e^{-k}(k^2 + k(x+2) + (x+2)(x+1)) \\ &= (x+2)(x+1)x\left[\gamma(x,k) - k^x e^{-k}\left(\frac{1}{x} + \frac{k}{x(x+1)} + \frac{k^2}{x(x+1)(x+2)}\right)\right]\end{aligned}$$

So, $\gamma(x,k) = \dfrac{\gamma(x+3,k)}{(x+2)(x+1)x} + k^x e^{-k}\left(\dfrac{1}{x} + \dfrac{k}{x(x+1)} + \dfrac{k^2}{x(x+1)(x+2)}\right)$

If this process for an arbitrary integer n is repeated, then the following equation will be gotten:

$$\begin{aligned}\gamma(x,k) &= \frac{\gamma(x+n,k)}{(x+n-1)\cdots x} + k^x e^{-k}\left(\frac{1}{x} + \frac{k}{x(x+1)} + \frac{k^2}{x(x+1)(x+2)} + \cdots + \frac{k^{n-1}}{x\cdots(x+n-1)}\right) \\ &= \frac{\gamma(x+n,k)}{(x+n-1)\cdots x} + k^x e^{-k}\sum_{i=0}^{n-1}\frac{k^i}{x(x+1)\cdots(x+i)}\end{aligned}$$

Since $\gamma(x+n,k)$ is equal to a real number for all $n > 0$ and

$\lim_{n\to\infty}\left[(x+n-1)\cdots x\right] = \infty$, one can deduce that $\lim_{n\to\infty}\dfrac{\gamma(x+n,k)}{(x+n-1)\cdots x} = 0$.

Thus, as n goes to infinity $\gamma(x,k) = k^x e^{-k}\sum_{i=0}^{\infty}\dfrac{k^i}{x(x+1)\cdots(x+i)}$ ∴

Now it's the time to give the formula for case 3, namely the conversion of $\sum_{n=1}^{\infty} c_n$.

Theorem 10:

The series $\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} \frac{1}{(d+r)(2d+r)\cdots(dn+r)}$ converges to

$$\left(\frac{1}{d}\right)^{-r/d} e^{1/d} \cdot \gamma\left(1+\frac{r}{d}, \frac{1}{d}\right)$$

for $d \neq 0$.

Proof:

First of all, note that:

$$\begin{aligned}(d+r)(2d+r)\cdots(dn+r) &= d\left(1+\frac{r}{d}\right) d\left(2+\frac{r}{d}\right)+\cdots+d\left(n+\frac{r}{d}\right) \\ &= d^n\left(1+\frac{r}{d}\right)\cdots\left(n+\frac{r}{d}\right)\end{aligned}$$

So $\sum_{n=1}^{\infty} c_n$ can be rewritten as $\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} \frac{(1/d)^n}{\left(1+\frac{r}{d}\right)\cdots\left(n+\frac{r}{d}\right)}$.

Adding 1 to n leads to the equation,

$$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} \frac{(1/d)^n}{\left(1+\frac{r}{d}\right)\cdots\left(n+\frac{r}{d}\right)} = \frac{1}{d}\sum_{n=0}^{\infty} \frac{(1/d)^n}{\left(1+\frac{r}{d}\right)\cdots\left(n+1+\frac{r}{d}\right)}.$$

If $x = 1 + \frac{r}{d}$ and $k = \frac{1}{d}$, then by the Theorem 9, the following result will be:

$$\sum_{n=0}^{\infty} \frac{(1/d)^n}{\left(1+\frac{r}{d}\right)\cdots\left(n+1+\frac{r}{d}\right)} = \frac{\gamma\left(1+\frac{r}{d},\frac{1}{d}\right)}{\left(1/d\right)^{1+r/d} e^{-1/d}}$$

Thus, $\sum_{n=1}^{\infty} c_n = \frac{1}{d}\frac{\gamma\left(1+\frac{r}{d},\frac{1}{d}\right)}{\left(1/d\right)^{1+r/d} e^{-1/d}} = \left(\frac{1}{d}\right)^{-r/d} e^{1/d}\cdot\gamma\left(1+\frac{r}{d},\frac{1}{d}\right)$ ∴

Since $\Gamma(x) = \Gamma(x,k) + \gamma(x,k)$, note that Theorem 10 can be restated as:

$$\sum_{n=1}^{\infty} \frac{1}{(d+r)(2d+r)\cdots(dn+r)} = \left(\frac{1}{d}\right)^{-r/d} e^{1/d}\cdot\left[\Gamma\left(1+\frac{r}{d}\right) - \Gamma\left(1+\frac{r}{d},\frac{r}{d}\right)\right].$$

<u>Examples:</u>

a) $\sum_{n=1}^{\infty} \frac{1}{5\cdot 7\cdots(2n+3)} \qquad = \frac{1}{5} + \frac{1}{5\cdot 7} + \frac{1}{5\cdot 7\cdot 9} + \cdots$

$$= \left(\frac{1}{2}\right)^{-3/2}\sqrt{e}\left[\Gamma\left(1+\frac{3}{2}\right) - \Gamma\left(1+\frac{3}{2},\frac{1}{2}\right)\right]$$

$$= 2\sqrt{2e}\left[\frac{3\sqrt{\pi}}{4} - \Gamma\left(\frac{5}{2},\frac{1}{2}\right)\right]$$

$$\approx 0.23206$$

b) $\sum_{n=1}^{\infty} \frac{1}{4 \cdot 7 \cdots (3n+1)} = \frac{1}{4} + \frac{1}{4 \cdot 7} + \frac{1}{4 \cdot 7 \cdot 10} + \cdots$

$$= \left(\frac{1}{3}\right)^{-1/3} e^{1/3} \left[\gamma\left(\frac{4}{3}, \frac{1}{3}\right)\right]$$

$$\approx \sqrt[3]{3e}\,[0.14387]$$

$$\approx 0.28958$$

b) $\sum_{n=1}^{\infty} \frac{1}{(-5) \cdot (-2) \cdots (3n-8)} = \frac{1}{-5} + \frac{1}{(-5) \cdot (-2)} + \frac{1}{(-5) \cdot (-2) \cdot 1} + \cdots$

$$= \left(\frac{1}{3}\right)^{8/3} e^{1/3} \left[\gamma\left(\frac{-5}{3}, \frac{1}{3}\right)\right]$$

$$\approx \sqrt[3]{\frac{e}{3^8}}\,[0.38844]$$

$$\approx 0.02896$$

Conclusion:

In conclusion, in this investigation, a way to make the example I took from my book into a generalized form was made. In order to do that, two different series were made and found the convergence of two different series afterward. The research question of this investigation was "Is it possible to obtain the formulas for the convergence of the series defined by multiplications of consecutive terms of an arithmetic sequence in a reciprocal form by using case-by-case proof?". According to the result of this investigation, it has been seen that it is possible to find convergence for both series.

The series are defined as:

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{\prod_{k=n}^{m+n-1} a_k} \qquad \sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} \frac{1}{\prod_{k=1}^{n} a_k}$$

In both series, case-by-case proof took place in 3 stages. First, it was looked for $\{a_n\} = \{dn\}$, then it was looked for $\{a_n\} = \{n + r\}$. Finally, $\{a_n\} = \{dn + r\}$ where the first two cases are combined. The final states of the two series for $\{a_n\} = \{dn + r\}$ are as follows:

For $\sum_{n=1}^{\infty} b_n$, it is equal to $\frac{1}{d(m-1)} \frac{1}{(d+r)(2d+r)\cdots(d(m-1)+r)}, m > 1$ where $\{a_n\} = \{dn + r\}$ is a non-zero sequence and d is a non-zero real.

For $\sum_{n=1}^{\infty} c_n$, it is equal to $\left(\frac{1}{d}\right)^{-r/d} e^{1/d} \cdot \gamma\left(1 + \frac{r}{d}, \frac{1}{d}\right), d \neq 0$

The limitation of this study may be the workload of doing the proofs all by myself and searching for alternative versions of functions to make the proof more easier. This is a strong foundation for finding convergence formulas, but it is a difficult task in terms of processing overhead. Therefore, it is thought that if computer analysis programs such as Wolfram Alpha were used to find the place where the series converges, the workload could have been reduced.

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