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Investigation of Taylor Series Remainders

Mathematics Extended Essay

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Research Question:

An investigation into Taylor series and investigating how close they are to the original function.

1. INTRODUCTION

Functions are one of the most used aspects of mathematics. It lets us calculate, represent and approximate lots of different results. One of their most important usages is how they can represent real-life problems. For example a car's speed can be represented as a linear function in the form ax + b, which is a easy to understand function and can be used with ease. But some functions are not so easy to deal with and compute. For these functions, we may use Taylor series representations.

For example, they are used in calculators. Humble calculators can only do our basic operations of addition, subtraction, multiplication and division. How are they going to calculate special functions like cosh or just normal trigonometric functions, sine and cosine? Because all of these functions can be represented as an infinite series of these four main operations, which allows us to calculate these functions.

The first step to understand Taylor series are Taylor polynomials, which gives an approximation of a *k*-times differentiable function around a given point by a polynomial of degree *k*, called the *k*th-order Taylor polynomial. There are several versions of Taylor's theorem, some giving explicit estimates of the approximation error of the function by its Taylor polynomial. They are named after the mathematician Brook Taylor who stated a version of it in 1715.

These polynomials are used to get simple polynomial representations of complex functions. It is fundamental in various areas of mathematics, physics and computer science. The "quality" of these estimations, in other words how close is it to the original functions, is the key factor that allows these series to be useful. But how can we determine how close this estimation is it? Is there a certain limit to our approximations? This investigation aims to explore this question.

This topic was special and interesting for me since I was always intrigued by numerical methods within mathematics. I wondered how people without the virtue of calculators achieve such accuracy with complicated problems. The remainder terms were an extension of our lessons and I was instantly interested in the topic. The numerical methods inside the series and remainders are somewhat my introduction on the vast world of numeric analysis and calculations.

2. EXPLORATION

In the first part of this exploration, let's look at the nature of Taylor polynomials and how we can construct the infinite series to give us the Taylor series. Taylor's theorem gives an approximation of a *k*-times differentiable function around a given point by a polynomial of degree *k*, called the *k*th-order Taylor polynomial.

To better understand Taylor series let's investigate the graph of $f(x) = \cos(x)$

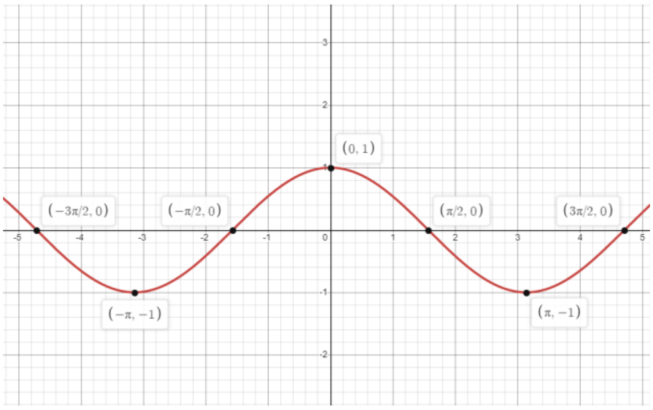

Figure 2.1: The graph of cos(x) with important values marked.

Suppose we want to approximate the value near $x = 0$ with a quadratic polynomial. Define $P(x)$ by:

$$P(x) = c_2x^2 + c_1x + c_0 \quad [1]$$

To start off, we want the polynomial to equal 1 at 0, since that is also true for cos(x). This is vital since this point will be the center of our estimation

So

$$\begin{aligned} P(0) &= c_2 0^2 + c_1 0 + c_0 \\ &= c_0 = 1 \end{aligned} \quad [2]$$

Choosing $c_0 = 1$ ensures the polynomial will equal 1 at x=0

Another mandatory point is that our polynomial should have the same tangent slope at the point we want to approximate, so it will not drift faster/slower from cos(x). This way we can ensure a better approximation.

We can take the derivative as

$$\frac{dcos}{dx}(0) = -\sin(x) = 0 \ [3]$$

So the tangent line is flat at cos(0)

$$\begin{gathered} \frac{dP}{dx} = 2c_2x + c_1 \\ P'(0) = \ c_1 \end{gathered} \quad [4]$$

This shows what we need to choose for c_1. This means c_1 has control over the first derivative of the graph around x=0. Whatever we assign c_2, the polynomials tangent at the point x=0 will be flat.

So, c_1 = 0

Our new polynomial is therefore:

$P(x) = 1 + 0x + c_2x^2$ or $P(x) = 1 + c_2x^2$ [5]

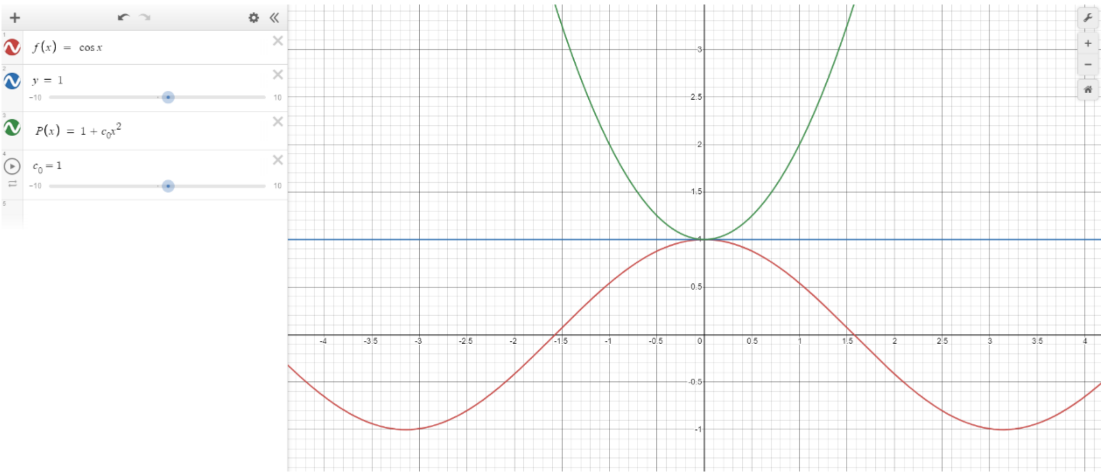

Figure 2.2: The graph of cos(x) and P(x) with the value of c1 set to 1.

The final thing to consider is how cos(x) has a negative rate of change after x = 0, in other words it is decreasing or has a negative 2nd derivative.

$$\frac{d^2\cos}{dx^2}(0) = -\cos(0) = -1 \ [6]$$

So we have to match this with our polynomial to make sure they curve at the same rate

$$\frac{d^2P}{dx^2}(x) = 2c_2 \text{ [7]}$$

To make this equal -1, we set c_2 at $-\frac{1}{2}$, which makes our final quadratic polynomial as:

$$\begin{aligned} P(x) &= 1 + 0x + (-\frac{1}{2})x^2 \\ &= 1 + (-\frac{1}{2})x^2 \end{aligned} \quad \text{[8]}$$

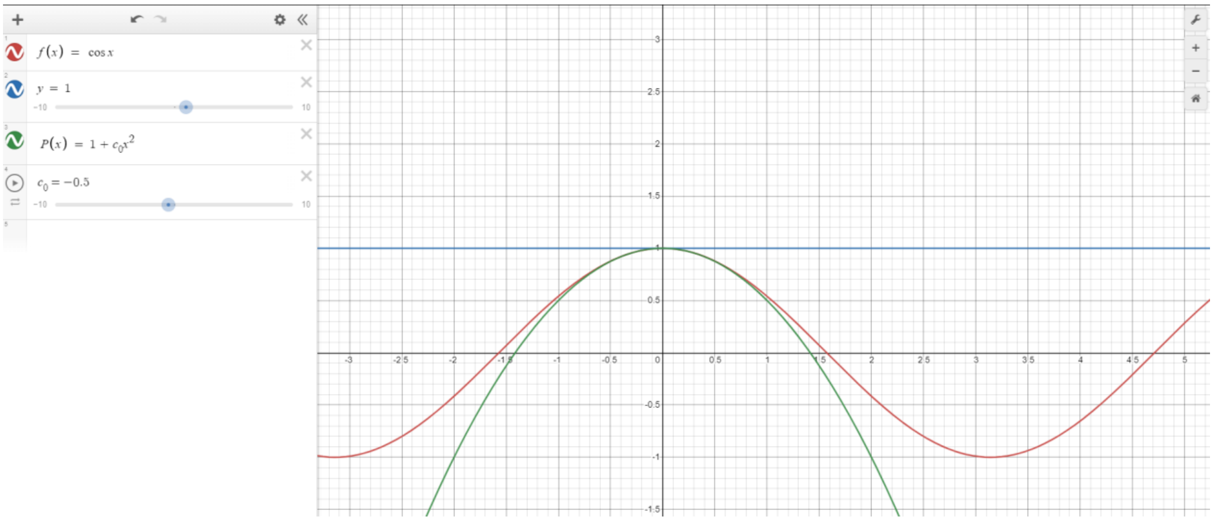

Figure 2.3: The graph of cos(x) and P(x) with the value of c1 set to -0.5.

For a quick test of this approximation, take $\cos(0.1)$

$P(0,1) = 1 - (-\frac{1}{2})(0.1)^2 \approx 0{,}995$ [9]

While the true value of cos(0.1) is ≈ 0.995004

But what if we added a cubic term which agrees with the 3rd derivative of $cosx$? What will it do to our estimation?

$$\frac{d^3\cos}{dx^3}(0) = \sin(0) = 0 \text{ [10]}$$

Then we add the 3rd order term to our polynomial as

$$
\begin{gathered}
P(x) = 1 + (-\frac{1}{2})x^2 + c_3x^3 \\
\frac{d^3P}{dx^3} = 6c_3 \qquad [11] \\
6c_3 = 0 \\
c_3 = 0
\end{gathered}
$$

So we see that the quadratic approximation for cos(x) is also the best cubic approximation

Let's continue and look at the 4th degree term, first

$$\frac{d^4\cos}{dx^4}(0) = \cos(0) = 1 \; [12]$$

Matching this value via the same method we get

$$
\begin{gathered}
P(x) = 1 - (-\frac{1}{2})x^2 + c_4x^4 \\
\frac{d^4P}{dx^4} = 24c_4 \\
24c_4 = 1 \qquad [13] \\
c_4 = \frac{1}{24}
\end{gathered}
$$

Which makes our final 4th order Taylor polynomial

$$P_4(x) = 1 - \left(-\frac{1}{2}\right)x^2 + \frac{1}{24}x^4 \; [14]$$

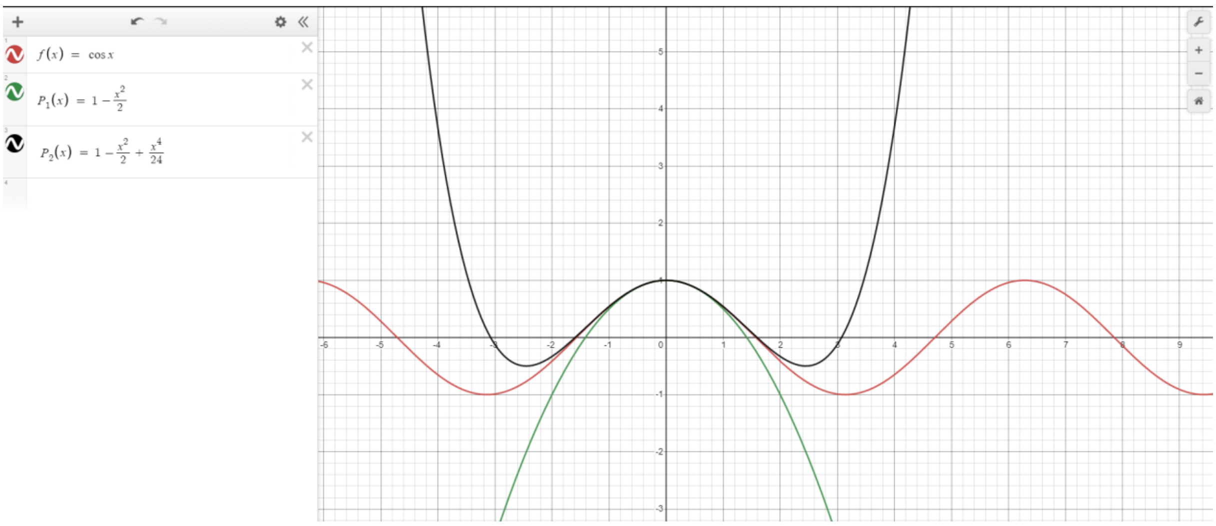

Figure 2.4: The graph of cos(x) and two polynomials constructed to fit the function

This gives us a better approximation. Take cos(0.1) again and see:

$P_4(0.1) \approx 0.995004166667$

While $\cos(0.1) \approx 0.995004165278$

When we investigate the nature of these polynomials we can realise a few things:

- The new higher order terms added don't interfere with the previous terms.
- The higher the order of the polynomial, the better the approximation will be.
- When taking n successive derivatives of x^n, by power rule we get n! as can be seen from the previous example where $\frac{d^4}{dx^4}(c_4x^4) = 1 * 2 * 3 * 4c_4x^0 = 24c_4$ so when we compute c_n , we have to divide by n!.
- As a conclusion, derivative about a point on our original graph gives us info about the output near that point.

To express our final Taylor polynomial for cos(x):

$$P(x) = 1 + (0\frac{x^1}{1!}) + (-1\frac{x^2}{2!}) + (0\frac{x^3}{3!}) + (1\frac{x^4}{4!})\ldots \quad [15]$$
$$P(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

This is the Taylor series for cos(x)

Which gets us to the general term used for Taylor polynomials:

<u>Definition:</u>

Let k $\geq$ 1 be an integer and let the function f: R → R be n times differentiable at the point a ∈ R. Then there exists a function h_k R → R such that

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(\mathrm{x}-\mathrm{a})^2 + \cdots$$

$$\frac{f^{(n)}(a)}{n!}(\mathrm{x}-a)^n + R_n(x)[16]$$ [1]

[1] 'Taylor's theorem'. Retrieved from: https://en.wikipedia.org/wiki/Taylor%27s_theorem

Where the function $R_n(x)$ is the remainder term of the polynomial, which closes the gap between the approximation and the original function. Which can be expressed as an power series to give us Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \quad [17]$$

Where $f^n(a)$ denotes the n^{th} derivative of f evaluated at point a.

This is the official definition of Taylor series. Taylor polynomials are the partial sums used to get an n degree polynomial.

If the Taylor series of a function is convergent, its sum is the limit of the infinite sequence of the Taylor polynomials. A function may differ from the sum of its Taylor series, even if its Taylor series is convergent. A function is analytic at a point x if it is equal to the sum of its Taylor series in some open interval containing x. This implies that the function is analytic at every point of the interval.

To better understand the concept of convergence and how it links to remainders, consider the following examples:

Taylor series for e^x

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad [18]$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$$

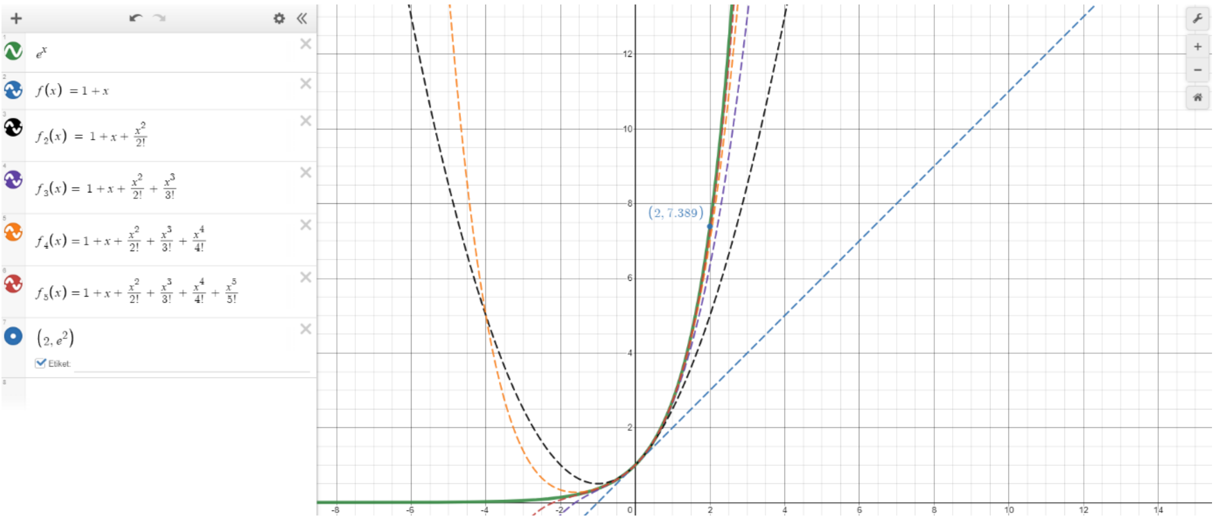

Figure 2.5: The graph of e^x with its Taylor polynomials of varying degrees.

Looking at the graphs of higher order Taylor polynomials, we can see that the polynomials with bigger terms are closer to the original function. We can also state this algebraically by taking the limit as n goes to infinity of the series:

$$\lim_{n\to\infty} \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad [19]$$

The factorial on the denominator gets bigger than the polynomial on the numerator. So, every consecutive gets smaller than the previous one. So the result of this limit is 0. It converges for any nonzero x.

if we plug in any number for x, say x = 2, the series will converges to e^2

We may even say the series “equals” e^x

Taylor series for $\ln(x)$ but since $\ln(0)$ is undefined, we cannot select our center at 0. We will select our center as $x = 1$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{n} \quad [20]$$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \dots$$

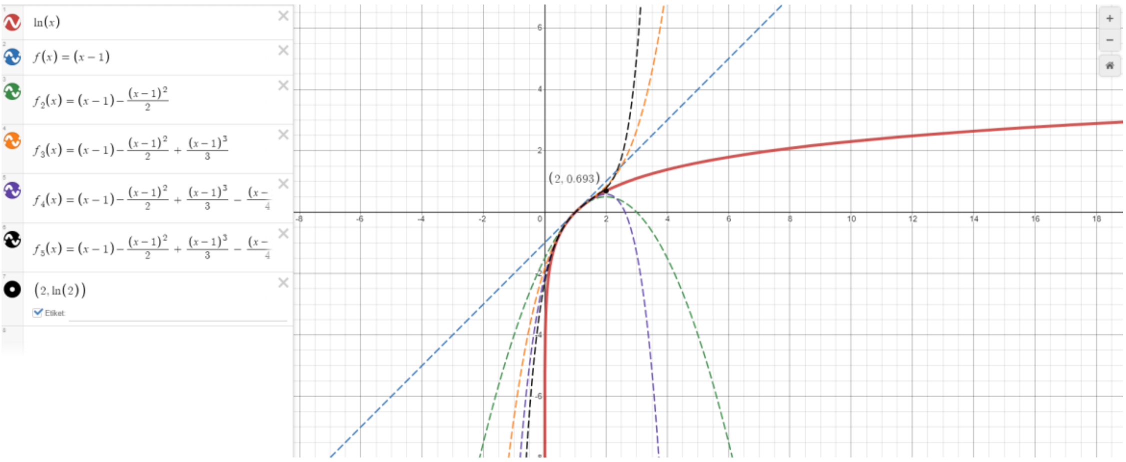

Figure 2.6: The graph of ln(x) with its Taylor polynomials of varying degrees.

As can be seen, the series gives increasingly better approximations, but only between a certain interval, which is$0 < x \leq 1$. Outside this range, the series fails to approach anything.

We can say it is “divergent”. There exists such a radius where the series does converge that can be found via the remainder theorem, which we will address later.

So, how can we find these error that may occur in approximations? For these, some remainder terms exists under certain condition that give us the quality of the approximation

3. REMAINDER TERMS

When considering Taylor polynomials, an error in the approximation is present, which can be reduced by adding higher order polynomial terms. The error in these approximations can also be expressed. The presence of this remainder term is crucial for us to understand how close the approximation is to the original function, as it can show the exact difference between the function and the approximation. This is the powerful tool that allows Taylor series to approximate functions to whatever we want. This why Taylor series work. It is that we can control the remainder to be as small as we want when calculating functions.

Consider the following function:

$$S(x) = \sum_{k=1}^{\infty} \frac{f^{(k)}(a)(x-a)^k}{k!} \; [21]$$

Which is our definition of Taylor series, where we can limit and divide the sum into two pieces and define one to be the remainder. This remainder will represent the approximation of the function.

$$T_n(x) = \sum_{k=1}^{n} \frac{f^k(a)(x-a)^k}{k!} + R_n(x) = \sum_{k=n+1}^{\infty} \frac{f^{(k)}(a)(x-a)^k}{k!} \text{ [22]}$$

Where the goal is to make the remainder $R_n(x)$ is "small" as possible, as close to 0 as possible.

First, let's visualize the the remainder in the function e^x and its first order taylor polynomial $T_1(x) = 1 + x$ centered at 0

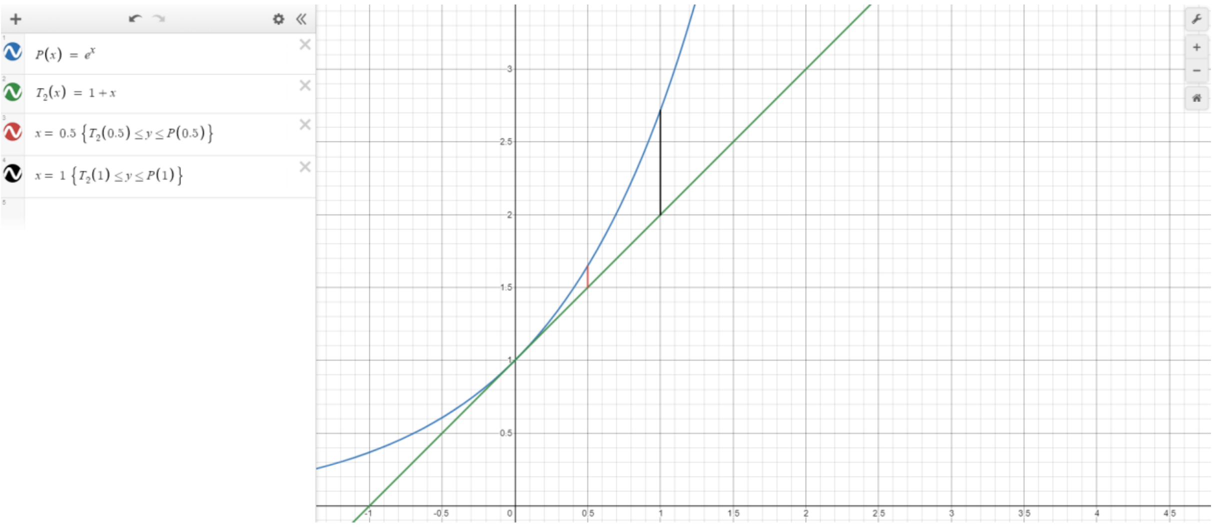

Figure 3.1: The graph of e^x and its first degree polynomial. The red line represents the difference at point x =0.5 and black line at x = 1

The red line represents the difference between the functions at the point x = 0.5 and the black line at x = 1. This shows us that, to no surprise, the remainder gets bigger and bigger as we move away from the center. This also shows us that the remainder depends on x, so it's a function of x. So the remainder term should depend on the difference between the point selected, a, and x.

Another observation can be made with linear approximations by the following functions:

$$f(x) = 1 + 0.5x^2 \text{ [23]}$$

$$M(x) = 1 + x^2 \text{ [24]}$$

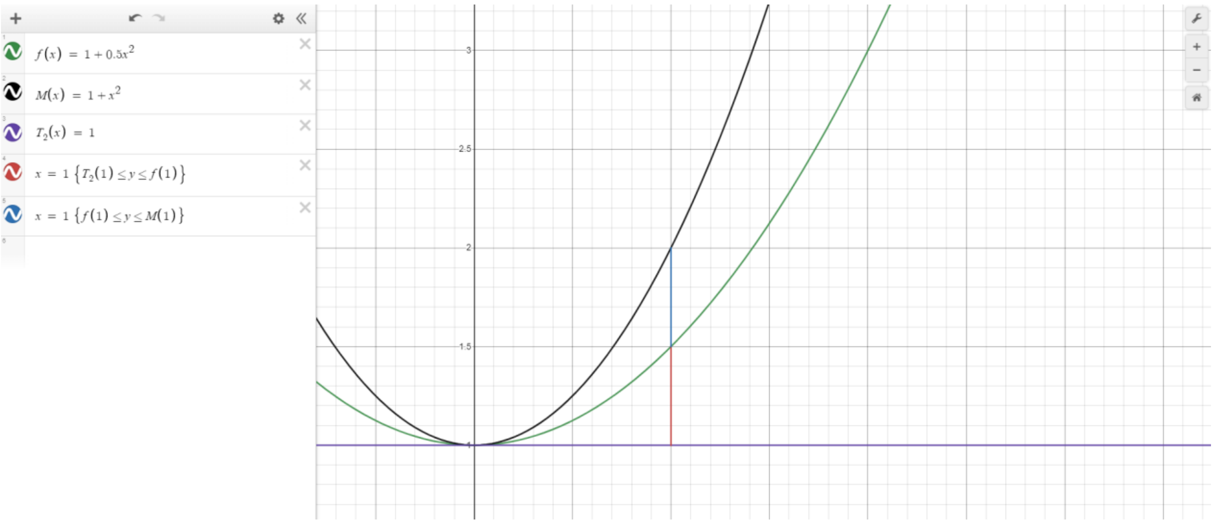

Figure 3.2: The graph of y = 1+0.5x² and y = 1+ x² and a linear approximation at the point x =1. The red line represents the difference between the approximation and the closest function, and blue line the other function.

Here, the functions P(x) and M(x) are quadratic functions with different openings, achieved by manipulating the coefficient of x^2. Here, the remainder between the 1st order Taylor approximation and the functions changes with the concavity, in other words, the second derivative of the functions.

So we can say the following inequality:

$$If\ \left|P^{(n+1)}(x)\right| \leq M, then\ R_n(x) \leq \frac{M}{(n+1)!}|x-a|^{n+1}\ for\ |x-a| \leq d\ [25]$$

Where we assume the (n+1)th derivative is bounded by some fixed value M, in some region defined by $|x-a| \leq d$, which the region can be changed as small as we need it to be.

So this depends on the distance x – a, on that bound on the (n+1)th derivative. This is a general form of the remainder term, which holds true for lower degrees of approximation.

Let’s consider the function P(x) = e^x again. Let’s consider the region where our approximation and error will be made as we did before: $|x| \leq d$

$$P(x) = e^x, |x| \leq d\ [26]$$

All order derivatives of P(x) will be the same, e^x. So we can say:

$$P^{(n+1)}(x) = e^x\ [27]$$

By the inequality we set before, our function has to be less than e^d. So e^x is some smaller value than that bound. That is the M value in the inequality. So we get the following where the “a” value is 0:

$$R_n(x) \leq \frac{e^d}{(n+1)!}|x|^{n+1} \text{ [28]}$$

To understand the nature of this expression, we know that we want to make the remainder as small as possible. To make it suit the function the best. Let's consider what happens when we add more and more terms to our Taylor series, thus increasing the number n. By this taking the limit as n goes to infinity, we get:

$$\lim_{n\to\infty} \frac{e^d\ |x|^{n+1}}{(n+1)!} \text{ [29]}$$

e^d is a constant, and the factorial on the denominator gets bigger than the polynomial on the numerator. So the value of this limit is 0. This means that if we take enough terms, the remainder can be as small as we wish, it will eventually become zero, thus no remainder. This can be used in all kinds of different applications. Depending on how much of error you want on your approximation. This shows the statement made before, where we can say that e^x is "equal" to its Taylor series for all x. This inequality is what makes the Taylor series so good of a tool to approximate functions as we please.

There are explicit formulas for the remainder term which are valid under some additional regularity assumptions on f. These enhanced versions of Taylor's theorem typically lead to uniform estimates for the approximation error in a small neighbourhood of the center of expansion, but the estimates do not necessarily hold for neighbourhoods which are too large, even if the function f is analytic.

Some examples on how the remainder term may be used is:

1. Estimate the error for a polynomial $P_k(x)$ of degree k estimating $f(x)$ on a given interval $(a - r, a + r)$.
2. Find the smallest degree k for which the polynomial $P_k(x)$ approximates $f(x)$ to within a given error tolerance on a given interval $(a - r, a + r)$.
3. Find the largest interval $(a - r, a + r)$ on which $P_k(x)$ approximates $f(x)$ to within a given error tolerance.

That means these information leads to:

4. When given the interval and degree, we can find the error involved.
5. When given the interval and error tolerance, we can find the degree of the polynomial.

6. When given the degree and error tolerance, we can find the interval of tolerance.

The most commonly used is the Langrange form of the remainder which reads:

Let f : R $\rightarrow$ R be k + 1times differentiable on the open interval with $f^{(k)}$ continuous on the closed interval between a and x. Then:

$$R_k(x) = \frac{f^{(k+1)}(\xi_L)}{(k+1)!}(x-a)^{k+1} \text{ [30]}$$

For some real number $\xi_L$between a and x

But a word of caution must be mentioned in this case: c is some number between a and x, and the formula doesn't specify what c might be and, in fact, c changes as x changes. What we know is that this c lies between a and x.

And also, for calculating integrals we can use:
Suppose f(x) is (n + 1)-times continuously differentiable. Then,

$$R_n(f)(x) = \int_c^x \frac{f^{n+1}(y)}{n!}(x-y)^n\,dy \text{ [31]}$$

4. ESTIMATES OF THE REMAINDER

While these formulas exist, it is usually much better to just estimate the remainder term just like the polynomial, which can be determined using inequalities deduced before.

$$|R_n(x)| \leq \frac{M|x-a|^{n+1}}{(n+1)!} \text{ [32]}$$

Where:
f(x) = the function we are approximating
a: the center of the series
n: the order of the approximating polynomial
x: We *choose* x to make $|x-a|^{n+1}$ as large as possible (to find an upper bound for the error)
$|f^{n+1}(x)|$: the absolute value of the (n+1)st derivative of the approximated function
M: this is the maximum value of $|f^{n+1}(x)|$on the given interval

This is rather useful, as for harder functions and more complex problems, finding ξ_L may not be possible to use the remainder theorem. This allows us to put an absolute maximum bound on our approximation, and assess if the approximation is within acceptable bounds of error. Then, we can adjust our original polynomial accordingly and reach a better conclusion.

5. APPLICATIONS

The applications of Taylor series are vast in many fields that include calculus. But to highlight how the approximations play a big role in the mathematical context, I would like to investigate 3 situations where Taylor series are the key to solving the problem.

1st: Proof of Euler's formula

Euler's formula is an important equation which links a lot of different concepts of mathematics into one simple equation. It is used in many different fields, especially in Physics.

$$e^{i\theta} = \cos(\theta) + isin(\theta) \text{ [33]}$$

A special case happens when x is equal to π which then reads:

$$e^{i\pi} = -1$$

$$e^{i\pi} + 1 = 0 \text{ [34]}$$

which is commonly referred as the "most beautiful equation in math". The proof of this formula comes from the power series definitions of the functions e^{θ}, $\sin x$ and $\cos x$. Which have the Taylor series centered at 0:

$$e^{\theta} = 1 + \frac{\theta}{2!} + \frac{\theta^2}{3!} + \frac{\theta^3}{4!} \ldots \text{ [35]}$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} \ldots \text{ [36]}$$

$$\cos\theta = \theta - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} \ldots \text{ [37]}$$

We know that the infinite series is equal to the functions as stated before, so we can assume these are true. When we add the sine and cosine functions, which looks kind of familiar to the e^{θ} function:

$$\sin\theta + \cos\theta = 1 + \theta - \frac{\theta^2}{2!} - \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{\theta^7}{7!} \ldots \text{ [38]}$$

But the alternating sign of the function is troublesome. When we investigate the function, we can see that we need such a number when squared gives -1, to make the terms equal in the equation. The number needed is then i, which is a number when squared to give -1. So we can add i to the e^{θ} to get:

$$1 + i\theta - \frac{(i\theta)^2}{2!} - \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} \ldots \text{ [39]}$$

Knowing that $i = \sqrt{-1}$, we can rewrite the equation as follows:

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} \dots \text{ [40]}$$

Then, we can group the terms that have an imaginary part and terms that do not. After that, we can factor out the i to get:

$$e^{i\theta} = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} \dots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} \dots\right) \text{ [41]}$$

Which are exact the terms of the cosine function and the sine function, multiplied by i. This gives us the final formula presented before:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \text{ [42]}$$

2nd: Evaluating non-elementary integrals

In calculus, some integrals do not have an elementary anti derivative, which means it cannot be written using a finite number of algebraic combinations or compositions of exponential, logarithmic, trigonometric, or power functions. This means we cannot apply standard integration techniques to solve them using the fundamental theorem of calculus. These integrals are called non-elementary integrals. These integrals still have useful and common applications in mathematics, which pushed people to solve them using other methods. One such integral is the integral of the normal distribution function

$$\frac{1}{\sqrt{\pi}}e^{-x^2}$$

$$\int \frac{1}{\sqrt{\pi}}e^{-x^2}\,dx \text{ [43]}$$

This integral arises often in applications in probability theory. The $\frac{1}{\sqrt{\pi}}$ outside comes from the result of the Gaussian integral, $\int_{-\infty}^{\infty} e^{-x^2}$. This ensures the area under the whole graph is 1, which is required for it to be valid probability density function. One way to evaluate such integrals is by expressing the integrand as a power series and integrating term by term.

First, the taylor series centered at 0 for $\frac{1}{\sqrt{\pi}}e^{-x^2}$ is given by:

$$\frac{1}{\sqrt{\pi}}e^{-x^2} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi}} \frac{(-x^2)^n}{n!} \text{ [44]}$$

Opening up the sum we get the terms as:

$$\frac{1}{\sqrt{\pi}} - \frac{1}{\sqrt{\pi}}x^2 + \frac{x^4}{\sqrt{\pi}\,2!} - \frac{x^6}{\sqrt{\pi}\,3!} + \cdots\ (-1)^n \frac{x^{2n}}{\sqrt{\pi}\,n!} + \cdots$$

Which can be written as:

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{\sqrt{\pi}\,n!}\ [45]$$

Therefore:

$$\frac{1}{\sqrt{\pi}} \int e^{-x^2}\,dx = \frac{1}{\sqrt{\pi}} \int \left(1 - x^2 + \frac{x^4}{2!} - \cdots\right) dx$$

$$= \frac{1}{\sqrt{\pi}}x - \frac{x^3}{\sqrt{\pi}3} + \frac{x^5}{\sqrt{\pi}2!\,.\,5} - \frac{x^7}{\sqrt{\pi}\,3!\,.\,7} + \ \ldots\ + (-1)^n \frac{x^{2n+1}}{\sqrt{\pi}(2n+1)n!} + \cdots + c\ [46]$$

Where c is the arbitrary constant of the indefinite integral.

Let's use the first three terms, $P_4(x)$, to estimate the value of the original function on the bounds 0 to 1. Plus we will have the remainder, $R_4(x)$. This will get us the following integral.

$$\frac{1}{\sqrt{\pi}} \int_0^1 e^{-x^2}\,dx = \frac{1}{\sqrt{\pi}} \int_0^1 P_4(x) + R_4(x)\,dx\ \ [45]$$

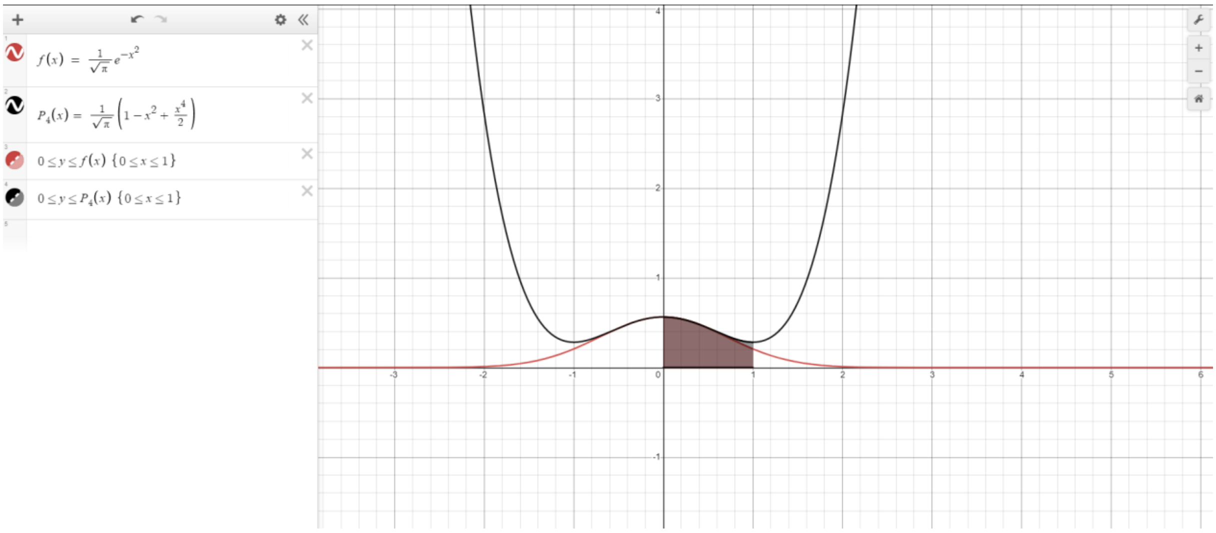

Figure 5.1: The graph of the normal distribution function and its second order Taylor series estimation. The black area represents the real result of the integral while red the approximation's integral.

$P_4(x)$, is a polynomial which we can easily compute. We also know the formula for the integral for of the remainder, which we can use to set an upper bound on the error involved with the approximation. With the theorem we can see that

$$R_4\,(e^{-2})(x)\,dx = \int_0^x (-1)^{4+1} \frac{e^{-y}}{2!} (x-y)^4\,dy\ [47]$$

However, we must realize that

$$R_4(x) = e^{-x^2} - P_4(x) = e^{-x^2} - \left(\sum_{k=0}^{4} \frac{(-x^2)^k}{k!}\right) = R_4\left(e^{-x^2}\right)(x^2) \text{ [48]}$$

Therefore, our integral is the following

$$R_4(x) = \int_0^{x^2} (-1)^{4+1} \frac{e^{-y}}{4!} (x^2 - y)^4 \, dy \text{ [49]}$$

Unfortunately, this is not something we can easily integrate. However, we are not interested in the actual value of the integral. We are only interested in making this integral close to 0. How do we bound $R_2(x)$?

First, see that

$$y \in [0, x^2], e^{-y} \leq e^0 = 1 \text{ [50]}$$

And

$$(x^2 - y)^4 \leq (x^2 - 0)^4 = x^8 \text{ [51]}$$

Note also for all $y \in [0, x^2]$, we have

$$\frac{2^{-y}}{4!}(x^2 - y)^4 \geq 0 \text{ [52]}$$

These bounds are to maximize the possible error involved with the calculation, so we can determine if the worst case scenario is useful to the necessary calculation. It also eliminates the y values that are hard to calculate after the integral. So we can rewrite the integral from before as

$$\int_0^{x^2} \frac{e^{-y}}{4!} (x^2 - y)^4 \, dy \text{ [53]}$$

$$R_4(x) \leq \int_0^{x^2} \frac{1}{4!} x^8 \, dy \text{ [54]}$$

With the fundamental theorem of calculus, we can deduce this integral as

$$\left[\frac{1}{4!} x^8 y\right]_0^{x^2} = \frac{1}{24} x^{16} \text{ [55]}$$

Getting back to the original integral, [51]

$$\left|\int_0^1 R_4(x)\,dx\right| \le \left[\frac{1}{24}x^{16}\,dx\right]_0^1$$

$$= \frac{1}{24} \approx 4.1\,.\,10^{-2} \text{ [57]}$$

So we can see that the maximum error our approximation will be $4.1\,.\ 10^{-2}$. This is an useful thing to know for our integral, as we can make sure that the approximation will be within an acceptable bound. Let's calculate the 4thdegree polynomial to get the full value. The polynomial integral is written as follows:

$$\frac{1}{\sqrt{\pi}}\int_0^1 1 - x^2 + \frac{x^4}{2!}\,dx \text{ [58]}$$

One final thing to consider is the value of π. Due to us being in the modern age of technology and pretty much unlimited computing power, we can approximate π up to millions of digits. But the same can be said to this integral. We can use our near unlimited computing power to approximate the integral using numerical methods in the calculator. So to avoid circular reasoning about setting bounds for our integrals, we can also bound π as 3, which will be sufficient in our estimation.

$$= \frac{1}{\sqrt{3}}\left[x - \frac{x^3}{3} + \frac{x^5}{5 * 2!}\right]_0^1 = \frac{1}{\sqrt{3}}\left(1 - \frac{1}{3} + \frac{23}{30}\right) \approx 0.44 \text{ [59]}$$

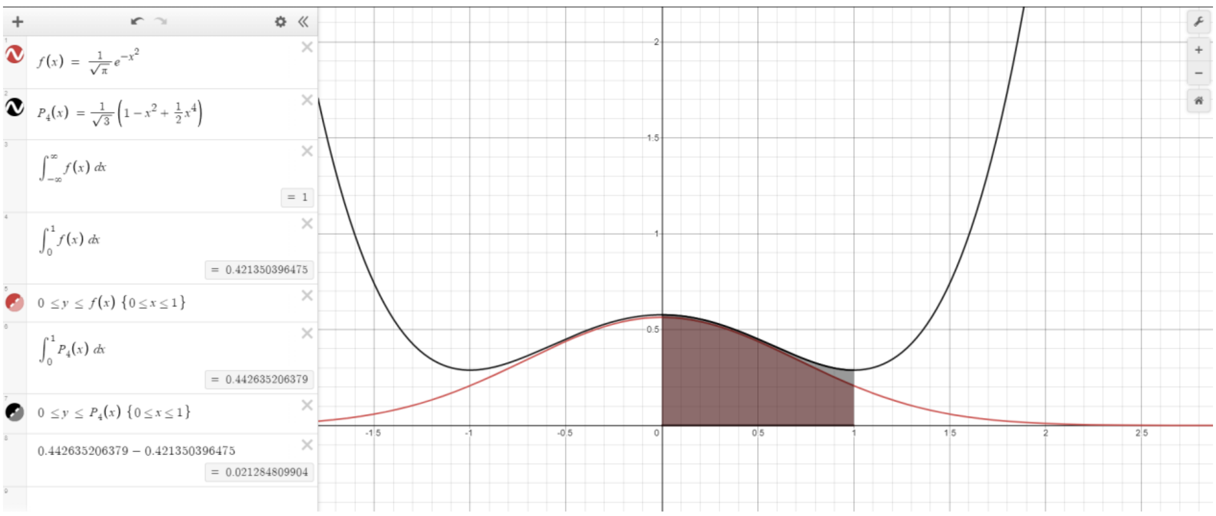

Figure 5.2: The graph of the normal distribution function and its second order Taylor series estimation but with 3 instead of π. The results are also displayed.

The real value of the integral is about $0.42 * 10^{-2}$, and the real error between the approximated polynomial value and the original integral is $\approx 2.1 * 10^{-2}$, which sits comfortably behind our maximum bound for error.

3rd: Energy in a pendulum

One final application I would like to explore is the pendulum problem from physics, where Taylor series are used to approximate the value of cosine. The usual formula used to calculate potential energy due to gravity is

$$PE = mgh \; [60]$$

But in the case of a pendulum which has an oscillating movement, we typically express the value of h with the length of the pendulum, which is

$$PE = mgL(1 - \cos x)[61]$$ [2]

Usually, the second order Taylor polynomial is used, since for this equation to hold true, the angle of the oscillation must be small so the equation comes to

$$PE = mgL\left(1 - 1 + \frac{\theta^2}{2!}\right) = mgL\left(\frac{\theta^2}{2!} + R_2(x)\right) \; [62]$$

For this, I used the clock we have in our kitchen:

Figure 5.3: The clock in my kitchen with a leaf-shaped pendulum.

[2] Gary Garber's Blog. 'Energy in a Pendulum'. Retrieved from: https://blogs.bu.edu/ggarber/interlace/pendulum/energy-in-a-pendulum/

The "pendulum" part has a mass of 19.24 grams, a length of 15.5 cm and makes a maximum angle of 12 degrees or $\frac{\pi}{15}$ radians. Using this data on the formula we get the maximum potential energy as

$$PE_{max} = 19.24 * 10^{-3} * 9.81 * 15.5 * \left(10^{-2} * \frac{\left(\frac{\pi}{15}\right)^2}{2} + R_2(x) \right) \approx 6.41 * 10^{-4} \; [63]$$

Now, by calculating, or estimating, the value of the remainder we can deduce how accurate this approximation is. To use the remainder estimation Theorem, let's deduce our variables in the equation. Our interval will be $\left[0, \frac{\pi}{15}\right]$, since that's our center and the point of approximation

f(x) = $\cos x$

a: 0

n: 2

x: to make $|x - a|^{n+1}$ as large as possible, let x = $\frac{\pi}{15}$

$|f^{n+1}(x)|$: the $\frac{d^3}{dx^3}$ of $\cos x$ is $\sin x$

M: maximum value of $\sin x$ on the given interval is $\sin\left(\frac{\pi}{15}\right)$

But, calculating $\sin\left(\frac{\pi}{15}\right)$ is as troublesome as calculating $\cos\left(\frac{\pi}{15}\right)$. So to avoid circular reasoning we let M = 1, since the maximum value of $\sin x$ on all x is 1. We will also use 3 for the value of pi, as we did in the integral application. By this, we get the following equation:

$$R_2(x) \leq 1 * \frac{\left(\frac{3}{15}\right)^3}{3!} \approx 1.3 * 10^{-3} \; [68]$$

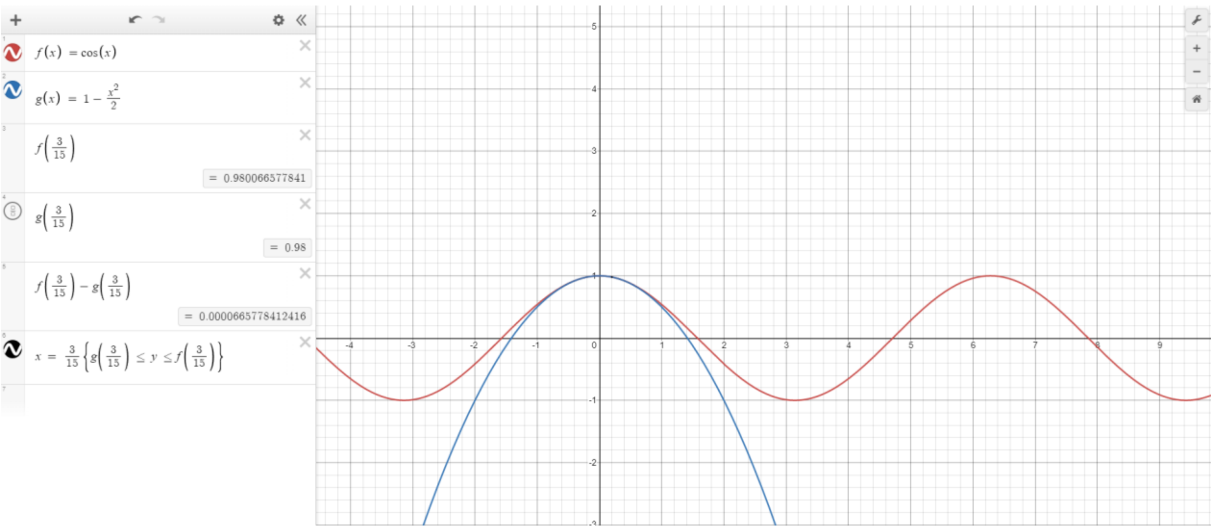

Figure 5.4: The graph of cos(x) and its second order Taylor approximation. The results of the functions are also shown..

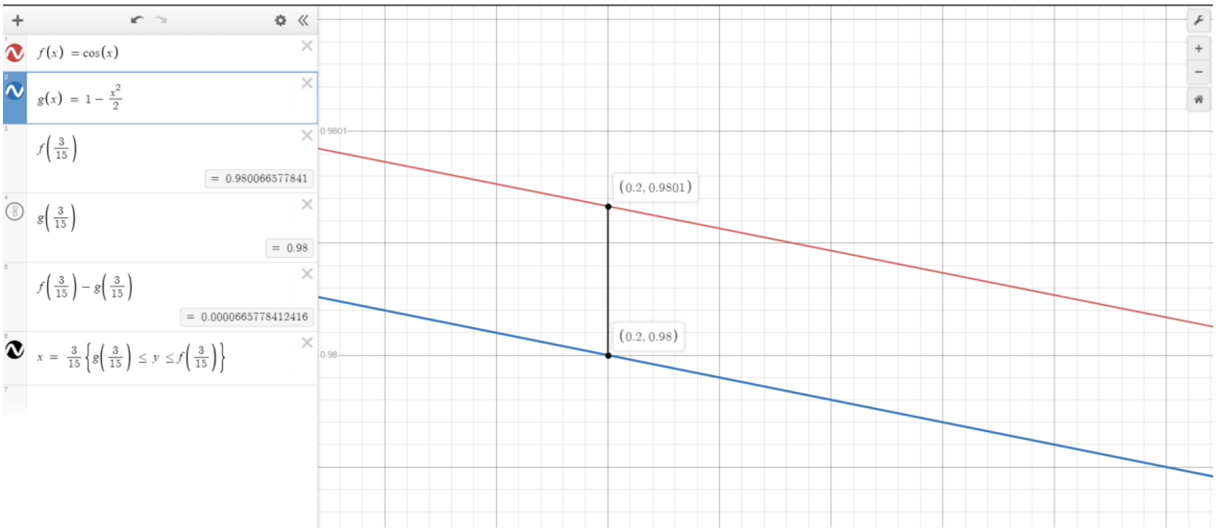

Figure 5.5: The difference between the functions is shown with the black bar.

Here, we can see that the real difference is about $8 * 10^{-4}$, which is unnoticeable in the graph unless extremely small intervals are used in the graph. The real error satisfies our inequality set from before. This difference can be argued to be negligible when calculating the potential energy in the pendulum. However, this negligibility depends on the context an on the tolerance of the situation.

6. CONCLUSION

The Taylor series are a powerful tool to approximate complicated equations and situations, especially in real life mathematics. They allow a precise degree of accuracy for computers and humans to calculate, since polynomials are much easier to work with. The nature of these polynomials allows a great deal of freedom for how close of an approximation to use, which makes it reliable and easy. There are many different fields and versions of these series which I didn't mention, such as their usage in complex analysis and in multivariable calculus. These all have great applications in advanced levels of mathematics.

Certain limitations do exist, such as the limitation of the function being infinitely differentiable. This makes it so that the series may not be used in every scenario. Another big limitation of the series is that the approximation cannot extrapolate the function for very long before rapidly diverging. There also exist the problem of convergence and divergence, where the series may give a wrong impression about. Some series, such as ln(x), have a limited interval of convergence. This makes it more complicated to use in certain scenarios. Also, other methods of approximation exist, which may offer different degrees of accuracy across a greater convergence, such as the Pade approximations. These offer a different method, which follow the function for a bigger radius of convergence.

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