Where is The Center of Mass of a Die Located?

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Finding The Center of Mass of a Die

Introduction

Humans have been rolling dice for thousands of years but up until the modern age dice have been produced in rough qualities. They had a habit of wearing out and becoming uneven. Naturally, the poor quality of a die defeats its whole purpose of randomness. However, modern technology has allowed the mass production of perfectly even and durable dice. Even so, can western type dice provide the pure randomness we have come to expect of them? Why is there a specific type of dice called “Casino Dice”? How does it differ from the traditional design? What influences the outcome of a roll?

Traditional Western Type Dice (1)

Casino Dice (2)

There are many occurrences that affect dice rolls, human throw, environmental conditions and the dice itself. What effects are derived by the dice itself? Why are casino dice visibly different than the traditional type of dice? Casino dice were invented to be fairer than normal dice. They are fairer because of their comparatively more evenly located centers of mass.

Traditional dice have engraved holes to serve as the pips representing the number on each face. Therefore, even though the cube may be perfectly even geometrically speaking, dice aren't exactly fair since there is an uneven distribution of mass. Whereas casino dice have pips painted on top of the cubes with even the paint's mass considered some of the time. This makes it as fair as possible. Also, casino dice are cast on tables with intricate design, dice always hit the ridge rubber to the sides which makes the dice spin unpredictably to prevent people from manipulating dice throws [(5)].

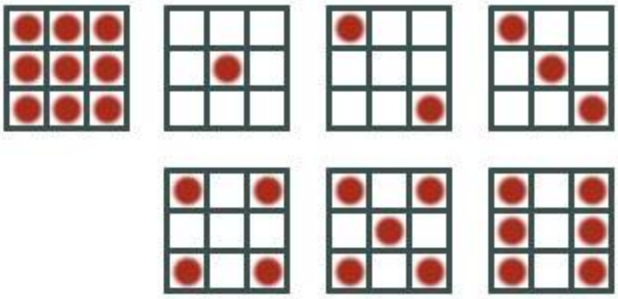
Methodology

General Idea

Are traditional dice really unbalanced? We can understand it better if we locate its center of mass. How can we locate the center of mass of such a seemingly irregular 3D shape? We must break it down to a simpler system of shapes. A cube is made up of 6 pyramids whose bases are the 6 faces of the cube. Then we must find the centers of mass' of each pyramid, treating them as a system of particles to solve for the whole die. Each pyramid can be simplified into a system of two pieces. The pyramid without engraved dots and the dots together. Due to the location of the dots in every face their common center of mass is the center of the base square, which is linear with the center of mass of the pyramid without dots, itself located on the height of the pyramid. This makes it so the two line up and share the same value on 2 of the 3 axes. This enables us to think of it as 2 points in space.

Dice Details

This will be based on the 25mm die since it is both common and sizable enough to be conclusive. It can be clear cut or round edged, this is unimportant it would be perfectly symmetrical either way. It is assumed that the die has constant density.



This image represents the faces of a die and the placements of dots. Each dot is half of a sphere. Diameters of the dots can be considered as 2r, the distance in between the dots r and the distance between a dot and the frame (side of the square) r/2 [3]. The sum of the r values adds up to 9r which is equal to the 2.5 cm sides of our 25 mm die. R = 0.2777777777777778 which we will round to R = 0.278 cm.

The die is accepted to be situated so that the six pyramids inside the cube starting from the center of the die expand in the x, y, z axis. Pyramid 1 expands in the -y direction, pyramid 2 in -x, pyramid 3 in -z, pyramid 4 in z, pyramid 5 in x and pyramid 6 in y. The pyramids are named after the number on their square base which is also the number of hemispherical carvings on their base, the number n.

Calculations

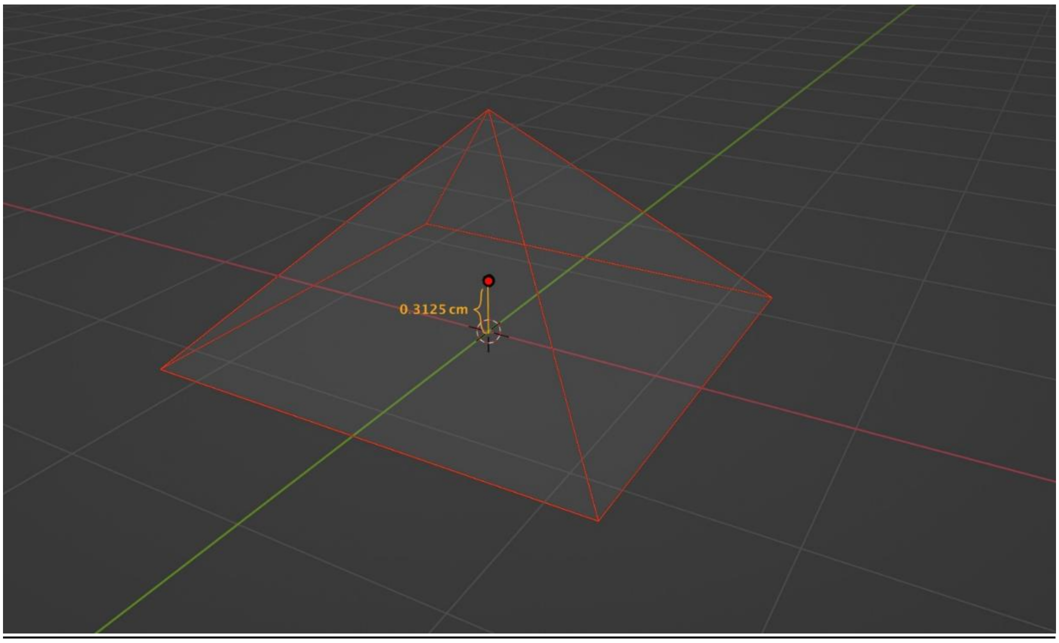
Formulas and Prerequisites

- Center of mass of a system of particles equation (for 2 objects): $x_{(com)} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$ [4]
- Density Formula: $\rho = \frac{m}{V}$

- Volume of a square based pyramid with sides of the square a, and height of the pyramid h: $V_p = \frac{a^2 h}{3}$
- The center of mass of a square based pyramid of radius r is aligned with the center of mass of the a-sided square base and $\frac{h}{4}$ upwards from the bottom.
- Volume of a hemisphere of radius r: $V_{hs} = \frac{2\pi r^3}{3}$
- The center of mass of a hemisphere of radius r is aligned with the center of the circle, located inside the hemisphere $\frac{3r}{8}$ from the circular base.

The center of mass of a system equation requires mass, however, we do not require mass. The die has constant density. Therefore, $\rho_1 = \frac{m_1}{V_1} = \rho_2 = \frac{m_2}{V_2}$, m_1 can be expressed as $m_1 = \frac{V_1}{V_2} m_2$. We can plug this expression in the center of mass equation and obtain G = $\frac{\frac{V_1}{V_2} m_2 x_1 \pm m_2 x_2}{\frac{V_1}{V_2} m_2 \pm m_2}$. The m_2 values cancel, mass is taken out of the equation leaving us with $\frac{\frac{V_1}{V_2} x_1 \pm x_2}{\frac{V_1}{V_2} \pm 1}$ expanding both the numerator and denominator with V_2, the equation ultimately becomes $\frac{V_1 x_1 \pm V_2 x_2}{V_1 \pm V_2}$. Even though neither mass nor volume can be negative quantities we can utilize the plus minus sign in the equation. When it has a minus in front, the lack of mass or volume can be thought of as something that doesn't necessarily have to be removed from one side. Instead of removing the mass/volume from one place, we can think of it as adding the same amount of mass/volume to the point on the other side that is the exact same distance away. This is not represented in the formula; however, it helps to see it that way rather than to mistake the minus sign as referring to negative mass or volume, both of which do not exist.

Calculations of Pyramids without Hemispheres

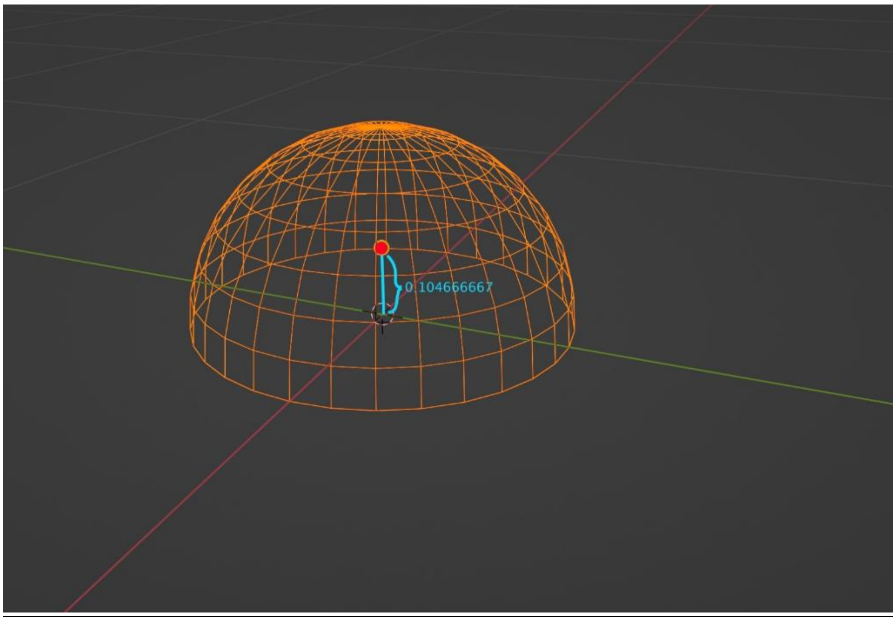


Centroid of a square based pyramid is h/4 above the base; thus 3h/4 is the distance between the top vertex, which is also the origin point, and the center of mass of the shape. A side of the die (a) is 25 mm, so a = 2.5 cm, h is the length between the origin (the center of the cube) and any edge of the cube. 2h = a, h = 1.25 $\frac{3h}{4} = \frac{3\times1.25}{4} = 0.9375 = x_1$

In order to find the true centers of mass of the pyramids, the location of the centroid of the pyramids without indentations and the hemisphere indentations will be used in the center of mass of a system of points equation as x_1 and x_2.

The volume is $V = \frac{a^2\times h}{3}$, using these values, we find $V = \frac{6.25\times1.25}{3} = 2.604166667$ for each pyramid without accounting for the hemisphere shaped indents.

Calculations of Hemispheres Shaped Indents

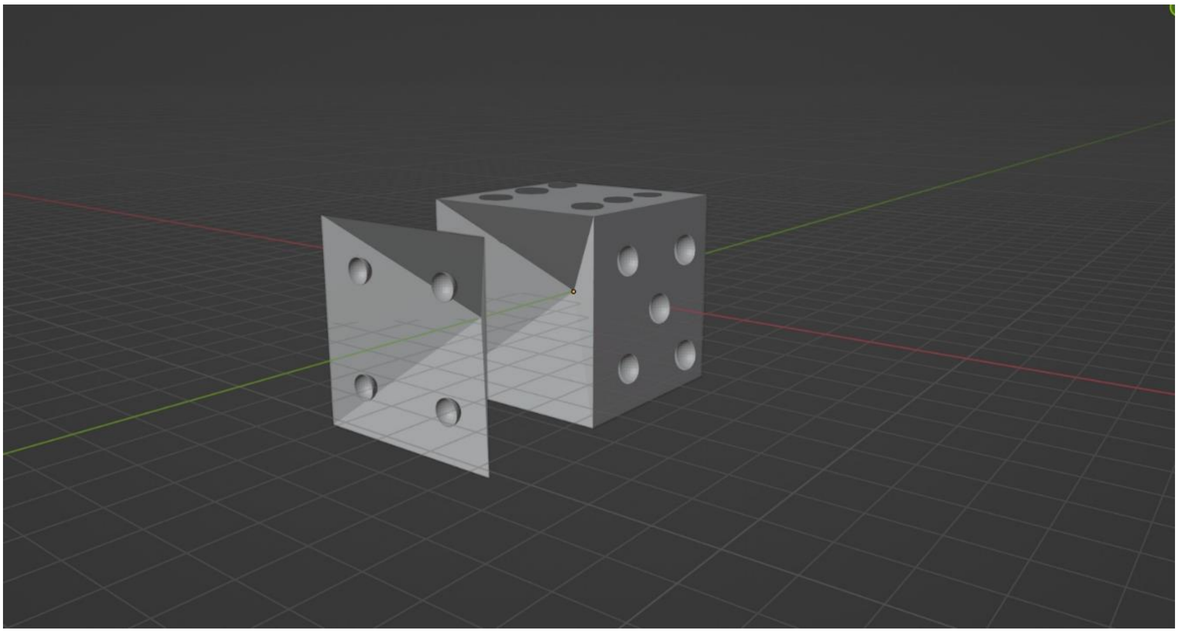


Centroid of a hemisphere is located 3r/8 deep from the circular base of the shape such that the line drawn from the centroid to the base of the circle lands perpendicular. r is the radius of the sphere and is the length from the circular base's center to any point on the curved surface.

$\frac{3r}{8} = \frac{3\times 0.2777777778}{8} = 0.1041666667$, circular bases of the hemispheres are a part of the pyramids' bases, so the centroid of the hemisphere is located $h - \frac{3r}{8} = 1.25 - 0.1041666667 = 1.145833333\ cm$ away from the origin point.

Calculations of Individual Pyramids

Volume of the indented pyramids (pyramids with dice values 1-6 carved on their bases) can be denoted as $V_{P_n} = V_P - n \times V_{hs}$, n is the number of that face, $n \times V_{hs}$ is the collective volume of all hemisphere indentions.



Pyramid 1:

$V_P = 2.604166667$, $1V_{hs} = 0.0448901556$, $x_1 = 0.9375$, $x_2 = 1.145833333$,

$$G_1 = \frac{V_1 x_1 \pm V_2 x_2}{V_1 \pm V_2} = \frac{V_P \times x_1 - nV_{hs} \times x_2}{V_P - nV_{hs}}$$

$$= \frac{2.604166667 \times 1.14 - 0.0448901556 \times 1.145833333}{2.604166667 - 0.0448901556} = 0.9338457971$$

$G_1 = (0, -0.9338457971, 0)$

Pyramid 2:

$V_P = 2.604166667$, $2V_{hs} = 0.0897803112$, $x_1 = 0.9375$, $x_2 = 1.145833333$,

$$G_2 = \frac{V_1 x_1 \pm V_2 x_2}{V_1 \pm V_2} = \frac{V_P \times x_1 - nV_{hs} \times x_2}{V_P - nV_{hs}}$$

$$= \frac{2.604166667 \times 1.14 - 0.0897803112 \times 1.145833333}{2.604166667 - 0.0897803112} = 0.9300611148$$

$G_2 = (-0.9300611148, 0, 0)$

Pyramid 3:

$V_P = 2.604166667$, $3V_{hs} = 0.1346704668$, $x_1 = 0.9375$, $x_2 = 1.145833333$,

$$G_3 = \frac{V_1 x_1 \pm V_2 x_2}{V_1 \pm V_2} = \frac{V_P \times x_1 - nV_{hs} \times x_2}{V_P - nV_{hs}}$$

$$= \frac{2.604166667 \times 1.14 - 0.1346704668 \times 1.145833333}{2.604166667 - 0.1346704668} = 0.9261388375$$

$G_3 = (0,0,-0.9261388375)$

Pyramid 4:

$V_P = 2.604166667$, $4V_{hs} = 0.1795606224$, $x_1 = 0.9375$, $x_2 = 1.145833333$,

$$G_4 = \frac{V_1 x_1 \pm V_2 x_2}{V_1 \pm V_2} = \frac{V_P \times x_1 - nV_{hs} \times x_2}{V_P - nV_{hs}}$$

$$= \frac{2.604166667 \times 1.14 - 0.1795606224 \times 1.145833333}{2.604166667 - 0.1795606224} = 0.922071323$$

$G_4 = (0,0,0.922071323)$

Pyramid 5:

$V_P = 2.604166667$, $5V_{hs} = 0.224450778$, $x_1 = 0.9375$, $x_2 = 1.145833333$,

$$G_5 = \frac{V_1 x_1 \pm V_2 x_2}{V_1 \pm V_2} = \frac{V_P \times x_1 - nV_{hs} \times x_2}{V_P - nV_{hs}}$$

$$= \frac{2.604166667 \times 1.14 - 0.224450778 \times 1.145833333}{2.604166667 - 0.224450778} = 0.9178503522$$

$G_5 = (0.9178503522,0,0)$

Pyramid 6:

$V_P = 2.604166667$, $6V_{hs} = 0.2693409336$, $x_1 = 0.9375$, $x_2 = 1.145833333$,

$$G_6 = \frac{V_1 x_1 \pm V_2 x_2}{V_1 \pm V_2} = \frac{V_P \times x_1 - nV_{hs} \times x_2}{V_P - nV_{hs}}$$

$$= \frac{2.604166667 \times 1.14 - 0.2693409336 \times 1.145833333}{2.604166667 - 0.2693409336} = 0.9134670739$$

$G_6 = (0{,}0.9134670739{,}0)$

Application of Vectors

Translating to Vectors

So far, the complexity of a dice's distinct shape has been reduced to 6 points in 3D space with only one value each on any of the 3 axes x, y, z. Even so, how could they be combined to show the exact location of the center of mass of an ideal die of constant density?

The 6 points all have different distances between them and the origin point. They also have unique values of volume attached to them. Since they can be represented in 3D space, have lengths (distance between them and the origin point) and magnitudes (volume), they can be regarded as vectors in space.

Reimagining these points to fit into this description is essential as it allows us to apply numerical operations on them and add them together in sum of vectors with ease.

$$G_1 \begin{pmatrix} 0 \\ -0.9338457971 \\ 0 \end{pmatrix} \qquad G_2 \begin{pmatrix} -0.9300611148 \\ 0 \\ 0 \end{pmatrix} \qquad G_3 \begin{pmatrix} 0 \\ 0 \\ -0.9261388375 \end{pmatrix}$$

$$G_4 \begin{pmatrix} 0 \\ 0 \\ 0.922071323 \end{pmatrix} \qquad G_5 \begin{pmatrix} 0.9178503522 \\ 0 \\ 0 \end{pmatrix} \qquad G_6 \begin{pmatrix} 0 \\ 0.9134670739 \\ 0 \end{pmatrix}$$

Scalar Multiplication with Vector Expressions

$V_{P1} = 2.559276511$ $V_{P2} = 2.514386356$ $V_{P3} = 2.4694962$

$V_{P4} = 2.4224606045$ $V_{P5} = 2.379715889$ $V_{P6} = 2.334825733$

$$G_1\begin{pmatrix} 0 \\ -0.9338457971 \\ 0 \end{pmatrix}.(2.55927651) = \overrightarrow{OG_1} = \begin{pmatrix} 0 \\ -2.389969613 \\ 0 \end{pmatrix}$$

$$G_2\begin{pmatrix} -0.9300611148 \\ 0 \\ 0 \end{pmatrix}.(2.514386356) = \overrightarrow{OG_2} = \begin{pmatrix} -2.338532977 \\ 0 \\ 0 \end{pmatrix}$$

$$G_3\begin{pmatrix} 0 \\ 0 \\ -0.9261388375 \end{pmatrix}.(2.4694962) = \overrightarrow{OG_3} = \begin{pmatrix} 0 \\ 0 \\ -2.28709634 \end{pmatrix}$$

$$G_4\begin{pmatrix} 0 \\ 0 \\ 0.922071323 \end{pmatrix}.(2.4224606045) = \overrightarrow{OG_4} = \begin{pmatrix} 0 \\ 0 \\ 2.235659704 \end{pmatrix}$$

$$G_5\begin{pmatrix} 0.9178503522 \\ 0 \\ 0 \end{pmatrix}.(2.379715889) = \overrightarrow{OG_5} = \begin{pmatrix} 2.184223067 \\ 0 \\ 0 \end{pmatrix}$$

$$G_6\begin{pmatrix} 0 \\ 0.9134670739 \\ 0 \end{pmatrix}.(2.334825733) = \overrightarrow{OG_6} = \begin{pmatrix} 0 \\ 2.13278643 \\ 0 \end{pmatrix}$$

Sum of Vectors

$$\overrightarrow{OG_1} + \overrightarrow{OG_2} + \overrightarrow{OG_3} + \overrightarrow{OG_4} + \overrightarrow{OG_5} + \overrightarrow{OG_6} = \begin{pmatrix} 0 \\ -2.389969613 \\ 0 \end{pmatrix} + \begin{pmatrix} -2.338532977 \\ 0 \\ 0 \end{pmatrix} +$$

$$\begin{pmatrix} 0 \\ 0 \\ -2.28709634 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 2.235659704 \end{pmatrix} + \begin{pmatrix} 2.184223067 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2.13278643 \\ 0 \end{pmatrix} = \vec{G}_{Die} =$$

$$\begin{pmatrix} -0.15430991 \\ -0.257183183 \\ -0.051436636 \end{pmatrix}$$

G_{Die} is the center of mass of the die as a whole and its coordinates are $\begin{pmatrix} -0.15430991 \\ -0.257183183 \\ -0.051436636 \end{pmatrix}$, ultimately this means that the die's center of mass exists closest the these faces in order 1,2,3,4,5,6. It is somewhat gravitating towards the faces 1, 2 and 3.

Conclusion

We have reached the conclusion that due to its exact coordinates $\begin{pmatrix} -0.15430991 \\ -0.257183183 \\ -0.051436636 \end{pmatrix}$, the center of mass of the die is closer to the faces 1, 2 and 3. Which means when the die is cast, Those three faces are more likely to end up on the bottom of the die, causing the roll to result in 6, 5 or 4 more frequently than 1, 2 and 3. These deviations from the geometric center of the cube, by approximately 0.05, 0.15 or 0.25 cm are not to be taken lightly. Considering each side of the cube is 2.5 cm to begin with, these values show some serious tilting.

This no doubt contributes to the dice we cast every day, even if it mitigated by random errors, or systematic errors or environmental conditions. In fact, some studies have delved into this topic questioning the same fact and whether it creates bias and unfairness. Some methods include thousands of dice rolls by hand and some utilize simulations that take the physical implications of the location of the center of mass into consideration.

In both instances, it is apparent that especially in smaller sample sizes, 4,5 and 6 are more likely to come up than 1, 2 and 3. In continuation of the experiment however, it is found that, the results tend to fit in a uniform distribution, that is a probability distribution with equally likely outcomes. So surely, many possible error causes and environmental effects are nullified by the sheer number of trials completed. This may also be used to explain why even though it is more likely than the other face values results, we do not always roll sixes.

Bibliography

1) https://en.wikipedia.org/wiki/File:6sided_dice_(cropped).jpg
2) https://merlinswakefield.co.uk/genuine-las-vegas-casino-dice-23355-p.asp
3) Blender 3D: Noob to Pro/Dicing with Dept, *Wikibooks*, Retrieved from https://en.wikibooks.org/wiki/Blender_3D:_Noob_to_Pro/Dicing_With_Depth
4) Center of Mass of System of Particles, *Physics Catalyst*, Retrieved from https://physicscatalyst.com/mech/center-of-mass.php
5) Fair Dice (Part 2) – Numberphile, *YouTube*, Retrieved from https://www.youtube.com/watch?v=8UUPlImm0dM