International Baccalaurate

Mathematics Higher Level

Extended Essay

Research Question: How can Stirling Numbers of the Second Kind be used in inductive and deductive mathematical proofs and used in real life problems?

Word Count: 3153

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Rationale

I am fond of Mathematics since I was a little child, when my teacher told me the story of Gauss in elementary school. I was amazed with Gauss' logical and simple way of thinking, and being able to find an answer to any problem in real life with Mathematics seemed as incredible as having a superpower. While growing up, I became particularly interested in Economics, and as I researched and read books about investments and stock markets, I realized that Mathematics was gradually becoming a great part of my reading material. Because of this, I chose Mathematics HL when I entered the IB in order to become familiar with more complicated concepts. As a result of my special interest in Maths and my ambition to study economics, I have chosen this extended essay to sharpen my skills and to help me adopt the refined yet complex approaches of mathematicians.

In middle school, the topic that I found the most interesting was the set theory because I like order and organization. Plus, it can be applied to virtually every other area, from history to physics. Also, I am fascinated with combinatorics. The idea of precisely knowing seemingly random chances and minimalizing risks interests me greatly, especially because I want to become an investor in the future and risk is a determining factor in economics. After some research, I have found that these two areas intersected in the topic Stirling Numbers, and the proofs and uses of these numbers seemed very exciting to study. With its high level of applicability in real life situations and curiosity-sparking complexity, I have chosen Stirling Numbers of the Second Kind as my Maths Extended Essay topic.

Historical Background and Definitions

What are Stirling Numbers?

Stirling Numbers consist of two separate and advanced functions. They are used in the solutions of problems regarding combination, permutation and set theory.

Stirling Numbers of the First Kind is basically advanced permutation, while Stirling Numbers of the Second Kind is an advanced combination function. In this extended essay, Stirling Numbers of the Second Kind will be explored.

What are the formulas?

$s(\Phi, \Omega) = (-1)^{(\Phi-\Omega)}. |s(\Phi, \Omega)|$

This function defines the Stirling Numbers of the First Kind. They have signs in front of them.

$$S(ß, \Phi) = \left(\frac{1}{\Phi!}\right) \sum_{\Omega=0}^{(\Phi-1)} (-1)^{\Omega} \binom{\Phi}{\Omega} (\Phi - \Omega)^{ß}$$

The function above is the function that defines Stirling Numbers of the Second Kind. They do not have signs in front of them.

What does the Stirling Numbers of the Second Kind function mean?

$$S(ß, \Phi) = \left(\frac{1}{\Phi!}\right) \cdot \sum_{\Omega=0}^{(\Phi-1)} (-1)^{\Omega} \cdot \binom{\Phi}{\Omega} \cdot (\Phi - \Omega)^{ß}$$

Stirling Numbers of the Second Kind is a function that connects the subset theory and combination. It was created while trying to find a formula for calculating the number of the different ways of partitioning ß elements to Φ groups without leaving any groups empty.

Q is the first set, and $|Q| = ß$. J is the second set, and $|J| = \Phi$.

At its simplest, they are defined as: $\frac{\text{The Number of Onto Functions}}{\Phi!}$

In a simple explanation, it solves the following problem: “In how many different ways can ß different objects be put into Φ different identical containers without leaving any containers empty?”

This is not an example that fully reflects the reality, but it is the most understandable one.

What do the Stirling Numbers of The Second Kind mean?

The function consists of the following elements:

$$\binom{\Phi}{\Omega}$$

Meaning: It is the shorthand notation of $\frac{\Phi!}{\Omega!\cdot(\Phi-\Omega)!}$ This is the formula to find the Ω element subsets of a set with Φ elements. This is a combination function, and it is the same as $\binom{\Phi}{\Phi-\Omega}$ because $\frac{\Phi!}{(\Phi-\Omega)!\cdot(\Phi-(\Phi-\Omega))!} = \frac{\Phi!}{\Omega!\cdot(\Phi-\Omega)!}$. This takes the unselected combinations from set A to set B into consideration, such as the subests containing 2, 3 or 4 elements etc.

$$\left(\frac{1}{\Phi!}\right)$$

Meaning: It is the principle used in finding how many different meaningful or meaningless words can be constructed from the letters of “algebra”. As there are two identical “a”s, the words created by swapping the places of the first and last “a” are the same, and they need to be eliminated. This is done by dividing the solution by 2! As we define the Φ containers to be identical, we count the same sequence many times. $\left(\frac{1}{\Phi!}\right)$ is used to eliminate the many identical sequences.

$$(\Phi - \Omega)^{ß}$$

Meaning: $\Phi^{ß}$ is the number of functions from set A to set B when $|A| = ß$ and $|B| = \Phi$.

For example, $A = \{C, D\}$ and $B = \{E\}$. $|A| = ß = 2$ and $|B| = \Phi = 1$.

Different combinations:

$C \rightarrow E,\ D \rightarrow E$. Number of functions from set A to set B is $1^2 = 1$.

If $A = \{C, D, G\}$ and $B = \{E, F\}$. $|A| = ß = 3$ and $|B| = \Phi = 2$.

Different combinations:

$C \rightarrow E,\ D \rightarrow E,\ G \rightarrow F;\ C \rightarrow F,\ D \rightarrow F,\ G \rightarrow E;$

$C \rightarrow E,\ D \rightarrow E,\ G \rightarrow E;\ C \rightarrow F,\ D \rightarrow F,\ G \rightarrow F;$

$G \rightarrow E,\ D \rightarrow E,\ C \rightarrow F;\ G \rightarrow F,\ D \rightarrow F,\ C \rightarrow E;$

$C \rightarrow F,\ G \rightarrow F,\ D \rightarrow E;\ C \rightarrow E,\ G \rightarrow E,\ D \rightarrow F.$

Number of functions from set A to set B is $2^3 = 8$. As no elements must be left unmatched in the domain, ß is the determining element of the $\Phi^{ß}$ function. For each element in set A, there are Φ different choices to match in set B. And as there are ß elements in set A, ß choices can be made from Φ choices. $(\Phi - \Omega)^{ß}$ gives the number of unmatched elements in set B.

Note: This function is the representation of set partitions as sequences using combination. The following elements are related with the polynomial sequence expansion of this function.

$$\sum_{\Omega=0}^{(\Phi-1)}$$

Meaning: Using $(\Phi - 1)$ is no different than writing Φ. Ω operates like a counter. In the function, using Φ is impractical because it means matching none of set A's elements with those of set B. As a result, the function starts with matching every element of set A with one element of set B, which is leaving $(\Phi - 1)$ elements of set A empty.

$(-1)^{\Omega}$, meaning: When the function is defined using polynomial sequence expansion, this expression provides the alternating (+) and (-) signs of the elements. It comes from the inclusion – exclusion principle.

Extra point: When Ω = Φ, none of set B's elements are left unmatched.

Deduction and Induction Proofs using Stirling Numbers of the Second Kind

Lemma 1:

$$\Omega^{\Phi} = \sum_{ß \leq \Phi} \binom{\Omega}{ß} \cdot ß! \cdot S(\Phi, ß),\ \text{for } \Omega \geq \Phi$$

Proof: First, its meaning needs clarification.

Ω^{Φ} = In how many different ways can you separate Φ elements into Ω distinct groups.

Or simply, how many functions are there, that go from set A to set B, when set a has Φ elements and set B has Ω elements.

Let's look at it from another perspective:

$$\sum_{ß \leq \Phi} \binom{\Omega}{ß} \cdot ß! \cdot S(\Phi, ß)$$
$$= \binom{\Omega}{0} \cdot 0! \cdot S(\Phi, 0) + \binom{\Omega}{1} \cdot 1! \cdot S(\Phi, 1) + \binom{\Omega}{2} \cdot 2! \cdot S(\Phi, 2) + \cdots + \binom{\Omega}{\Phi} \cdot \Phi! \cdot S(\Phi, \Phi)$$

$\binom{\Omega}{ß} \cdot ß! \cdot S(\Phi, ß)$ – What does it mean?

We again have set A with Φ elements and set B with Ω elements. Our functions will be from set A to set B. We want functions that go to exactly ß of set B's elements.

Which means $|\{X \in B : \exists\, a \in A \text{ such that } f(a) = x\}| = ß$. Then, how many such functions exist?

First choose the ß elements in set B. We can make this choice in $\binom{\Omega}{ß}$ different ways. Then, we need to find how many onto functions that fulfil olur conditions exist.

As the first explanations in this essay show, this number is ß! . S(Φ, ß)

So

$$\binom{\Omega}{ß} \cdot ß! \cdot S(\Phi, ß)$$

Means: How many functions from set A to set B fill exactly ß elements in set B?

Then the number of functions that fill 0 elements in B is;

$$\binom{\Omega}{0} \cdot 0! \cdot S(\Phi, 0)$$

Then the number of functions that fill 1 elements in B is;

$$\binom{\Omega}{1} \cdot 1! \cdot S(\Phi, 1)$$

Then the number of functions that fill 2 elements in B is;

$$\binom{\Omega}{2} \cdot 2! \cdot S(\Phi, 2)$$

Then the number of functions that fill Φ elements in B is;

$$\binom{\Omega}{\Phi} \cdot \Phi! \cdot S(\Phi, \Phi)$$

You can't fill more than Φ elements in B, because set B has Φ elements.

So if we add up all these expressions, we find how many functions are there, that go from set A to set B. This was Ω^{Φ}.

So

$$\sum_{ß \leq \Phi} (\Omega, ß) \cdot ß! \cdot \mathrm{S}(\Phi, ß)$$
$$= (\Omega, 0) \cdot 0! \cdot S(\Phi, 0) + (\Omega, 1) \cdot 1! \cdot S(\Phi, 1) + (\Omega, 2) \cdot 2! \cdot S(\Phi, 2) + \cdots + (\Omega, \Phi) \cdot \Phi! \cdot S(\Phi, \Phi)$$

Proof by Deduction: I will continue my proof by using the deduction method to prove Lemma 2.

Lemma 2:

$$\sum_{1 \leq k \leq N} \Omega^{\Phi} = \sum_{1 \leq ß \leq \Phi} \left(\frac{1}{ß + 1}\right) \cdot \mathrm{s}(\Phi, ß) \cdot (\Psi + 1) \cdot \Psi \cdot (\Psi - 1) \cdot \ldots \cdot (\Psi - ß + 1), \text{for } \Psi \geq \Phi$$

Proof: So again we need to understand what these mean.

$$\sum_{1 \leq \Omega \leq \Psi} \Omega^{\Phi} = 1^{\Phi} + 2^{\Phi} + 3^{\Phi} + \cdots + \Psi^{\Phi}$$

which is the sum of the number of the Φ^{th} power of numbers from 1 to Ψ

$$\sum_{1 \leq ß \leq \Phi} \left(\frac{1}{ß + 1}\right) \cdot \mathrm{S}(\Phi, ß) \cdot (\Psi + 1) \cdot \Psi \cdot (\Psi - 1) \cdot \ldots \cdot (\Psi - ß + 1)$$
$$= \left(\frac{1}{2}\right) \cdot \mathrm{S}(\Phi, 1) \cdot (\Psi + 1) \cdot \Psi + \left(\frac{1}{3}\right) \cdot \mathrm{S}(\Phi, 2) \cdot (\Psi + 1) \cdot \Psi \cdot (\Psi - 1) + \left(\frac{1}{4}\right) \cdot \mathrm{S}(\Phi, 3) \cdot (\Psi + 1) \cdot \Psi \cdot (\Psi - 1) \cdot (\Psi - 2) + \ldots + \left(\frac{1}{(\Phi + 1)}\right) \cdot \mathrm{S}(\Phi, \Phi) \cdot (\Psi + 1) \cdot \Psi \cdot \ldots \cdot (\Psi - \Phi + 1)$$

We need another Lemma to go further with this proof, so I continue with Lemma 3.

Lemma 3:

$$\sum_{ß \leq \Omega \leq \Psi} \binom{\Omega}{ß} = \binom{\Psi + 1}{ß + 1}$$

Let’s prove Lemma 3.

Proof: Is $\binom{\Psi+1}{ß+1}$ equal to $\binom{ß}{ß} + \binom{ß+1}{ß} + \binom{ß+2}{ß} + \ldots + \binom{\Psi}{ß}$

First of all, $\binom{ß}{ß} = 1 = \binom{ß+1}{ß+1}$.

Also, $\binom{P}{r+1} + \binom{P}{r} = \binom{P+1}{r+1}$

$\binom{ß}{ß} + \binom{ß+1}{ß} + \binom{ß+2}{ß} + \ldots + \binom{\Psi}{ß} = \binom{ß+1}{ß+1} + \binom{ß+1}{ß} + \binom{ß+2}{ß} + \ldots + \binom{\Psi}{ß}$

$\binom{ß+1}{ß+1} + \binom{ß+1}{ß} = \binom{ß+2}{ß+1}$

and we keep going on adding up.

The result is $\binom{\Psi+1}{ß+1}$

So we can now go back to what we first wanted to prove, Lemma 2.

Proof of Lemma 2 continued:

We will use Lemma 1 to calculate $1^{\Phi} + 2^{\Phi} + 3^{\Phi} + \ldots + \Psi^{\Phi}$.

$1^{\Phi} = \binom{1}{0}. 0! . S(\Phi, 0) + \binom{1}{1}.1! . S(\Phi, 1)$

$2^{\Phi} = \binom{2}{0}. 0! . S(\Phi, 0) + \binom{2}{1}. 1! . S(\Phi, 1) + \binom{2}{2}. 2! . S(\Phi, 2)$

$3^{\Phi} = \binom{3}{0}. 0! . S(\Phi, 0) + \binom{3}{1}. 1! . S(\Phi, 1) + \binom{3}{2}. 2! . S(\Phi, 2) + \binom{3}{3}. 3! . S(\Phi, 3)$

$4^{\Phi} = \binom{4}{0}. 0! . S(\Phi, 0) + \binom{4}{1}. 1! . S(\Phi, 1) + \binom{4}{2}. 2! . S(\Phi, 2) + \binom{4}{3}. 3! . S(\Phi, 3) + \binom{4}{4}. 4! . S(\Phi, 4)$

$\Psi^{\Phi} = \binom{\Psi}{0}. 0! . S(\Phi, 0) + \binom{\Psi}{1}.1! . S(\Phi, 1) + \binom{\Psi}{2}. 2! . S(\Phi, 2) + \binom{\Psi}{3}. 3! .S(\Phi, 3) + \binom{\Psi}{4}. 4! . S(\Phi, 4) + \ldots + \binom{\Psi}{\Psi}. \Psi! . S(\Phi, \Psi)$

When we add all these up, we get $1^{\Phi} + 2^{\Phi} + 3^{\Phi} + \ldots + \Psi^{\Phi}$, which equals:

$\left(\binom{1}{0} + \binom{2}{0} + \cdots + \binom{\Psi}{0}\right). 0! .S(\Phi, 0) + \left(\binom{1}{1} + \binom{2}{1} + \cdots + \binom{\Psi}{1}\right). 1! . S(\Phi, 1) +\left(\binom{2}{2} + \binom{3}{2} + \cdots + \binom{\Psi}{2}\right). 2! .S(\Phi, 2) + \left(\binom{3}{3} + \binom{4}{3} + \cdots + \binom{\Psi}{3}\right).3!.S(\Phi, 3) +\ldots+ \binom{\Psi}{\Psi} . \Psi!\ \ S(\Phi, \Phi)$

In this equation, $\left(\binom{1}{0} + \binom{2}{0} + \ldots + \binom{\Psi}{0}\right). 0! . S(\Phi, 0) = 0$ as $S(\Phi, 0) = 0$

And from Lemma 3, we know that $\binom{1}{1} + \binom{2}{1} + \ldots + \binom{\Psi}{1} = \left(\frac{\binom{\Psi+1}{2}}{\binom{2}{2} + \binom{3}{2} + \ldots + \binom{\Psi}{2}}\right) = \binom{\Psi+1}{3}$ etc.

And some terms will be 0 since $S(\Phi, (\Phi + 1)) = 0$, $S(\Phi, (\Phi + 2)) = 0$, $S(\Phi, (\Phi + 3)) = 0 \ldots$

So compressing further we get:

$\binom{\Psi+1}{2} . 1! . S(\Phi, 1) + \binom{\Psi+1}{3} . 2! . S(\Phi, 2) + \binom{\Psi+1}{4} . 3! . S(\Phi, 3) + \ldots + \binom{\Psi+1}{\Phi+1}. \Phi! . S(\Phi, \Phi)$

$= (\frac{1}{2}) . S(\Phi, 1) . (\Psi + 1) . \Psi + (\frac{1}{3}) . S(\Phi, 2) . (\Psi + 1) . \Psi . (\Psi - 1) + (\frac{1}{4}) . S(\Phi, 3) . (\Psi + 1) . \Psi . (\Psi -1) . (\Psi -2) + (\frac{1}{\Phi+1}) . S(\Phi, \Phi) . (\Psi + 1) . \Psi . \ldots . (\Psi - ß + 1)$

Which equals

$$\sum_{1\leq ß\leq\Phi} \left(\frac{1}{ß+1}\right). s(\Phi, ß). (\Psi + 1). \Psi . (\Psi - 1). \ldots . (\Psi - ß + 1)$$

proving Lemma 2.

Wait, but why did I do all of these?

I want to use Lemma 2, which is the expression of the Φ^{th} power of numbers from 1 to Ψ in terms of Stirling Numbers. But our formula doesn't look neat. Still, it has an interesting use.

$1^1 + 2^1 + \ldots + \Psi^1 = \frac{\Psi .(\Psi +1)}{2}$

is found from adding $1 + 2 + \ldots + \Psi = A$ to $\Psi + (\Psi - 1) + \ldots + 1 = A$ to get $\Psi.(\Psi + 1) = 2A$.

But how do we know $1^2 + 2^2 + \ldots + \Psi^2$, $1^3 + 2^3 + \ldots + \Psi^3$, $1^4 + 2^4 + \ldots + \Psi^4$ and so on?

Well, let's find this using Lemma 2.

First, put 1 instead of n in Lemma 2.

$$1^1 + 2^1 + \cdots + \Psi^1 = \left(\frac{1}{2}\right) \cdot S(1,1) \cdot (\Psi + 1) \cdot \Psi = \frac{\Psi \cdot (\Psi + 1)}{2} \text{ as } S(1,1) = 1$$

$$1^2 + 2^2 + \cdots + \Psi^2 = \left(\frac{1}{2}\right) \cdot \Psi \cdot (\Psi + 1) + \left(\frac{1}{3}\right) \cdot (\Psi + 1) \cdot \Psi \cdot (\Psi - 1) = \Psi \cdot (\Psi + 1) \cdot \left(\left(\frac{1}{2}\right) + \left(\frac{\Psi - 1}{3}\right)\right)$$
$$= \frac{\Psi \cdot (\Psi + 1) \cdot (2\Psi + 1)}{6}$$

Doing the same for $1^3 + 2^3 + \ldots + \Psi^3$.

It equals;

$$\left(\frac{1}{2}\right) \cdot S(3,1) \cdot (\Psi + 1) \cdot \Psi + \left(\frac{1}{3}\right) \cdot S(3,2) \cdot (\Psi + 1) \cdot \Psi \cdot (\Psi - 1) + \left(\frac{1}{4}\right) \cdot S(3,3) \cdot (\Psi + 1) \cdot \Psi \cdot (\Psi - 1) \cdot (\Psi - 2)$$

$$= \frac{\Psi \cdot (\Psi + 1)}{2} + \Psi \cdot (\Psi + 1) \cdot (\Psi - 1) + \left(\frac{1}{4}\right) \cdot \Psi \cdot (\Psi - 1) \cdot (\Psi - 2) \cdot (\Psi + 1)$$

$$= \frac{\Psi \cdot (\Psi + 1)}{2} \cdot (2 + 4 \cdot (\Psi - 1) + (\Psi - 1) \cdot (\Psi - 2))$$

$$= \frac{\Psi \cdot (\Psi + 1)}{4} \cdot (2 + 4\Psi - 4 + \Psi^2 - 3\Psi + 2)$$

$$= \frac{\Psi \cdot (\Psi + 1)}{4} \cdot (\Psi^2 + \Psi)$$

$$= \left(\frac{\Psi \cdot (\Psi + 1)}{2}\right)^2$$

So, by using this we can find $1^3 + 2^3 + \ldots + \Psi^3$, $1^4 + 2^4 + \ldots + \Psi^4$ and so on.

Proof by Induction: Now I want to investigate how I can prove Lemma 2 with induction method.

Can I find the previous formula using induction?

Let's show

$$1^2 + 2^2 + \cdots + \Psi^2 = \frac{(\Psi + 1) \cdot (2\Psi + 1)}{6}$$

Induct Ψ.

For $\Psi = 1$;

$$1^2 = \frac{(1 \cdot 2 \cdot 3)}{6}$$

Assume that the formula is correct for $\Psi = \Omega$ to show that it is also correct for $\Psi = \Omega + 1$

$$1^2 + 2^2 + \cdots + \Omega^2 = \frac{\Omega \cdot (\Omega + 1) \cdot (2\Omega + 1)}{6} \text{ is assumed.}$$

$$1^2 + 2^2 + \cdots + \Omega^2 + (\Omega + 1)^2 = \frac{\Omega \cdot (\Omega + 1) \cdot (2\Omega + 1)}{6} + (\Omega + 1)^2$$

$$= (\Omega + 1) \cdot \left(\frac{\Omega \cdot (2\Omega + 1)}{6} + (\Omega + 1)\right)$$

$$= (\Omega + 1) \cdot \left(\frac{2\Omega^2 + \Omega + 6\Omega + 6}{6}\right)$$

$$= (\Omega + 1) \cdot \left(\frac{2\Omega^2 + 7\Omega + 6}{6}\right)$$

$$= \frac{(\Omega + 1) \cdot (\Omega + 2) \cdot (2\Omega + 3)}{6}$$

$$\frac{\left((\Omega + 1) \cdot \left((\Omega + 1) + 1\right) \cdot (2 \cdot (\Omega + 1) + 1)\right)}{6}$$

So the formula is also correct for $\Psi = \Omega + 1$

Let's also prove

$$1^3 + 2^3 + \cdots + \Psi^3 = \left(\frac{\Psi \cdot (\Psi + 1)}{2}\right)^2$$

For $\Psi = 1$, it equals

$$1^3 = \left(\frac{1 \cdot 2}{2}\right)^2$$

Assume it's correct for $\Psi = \Omega$, then let's show that it's correct for $\Psi = \Omega + 1$.

$$1^3 + 2^3 + \cdots + \Omega^3 + (\Omega + 1)^3$$

$$= \left(\frac{\Omega \cdot (\Omega + 1)}{2}\right)^2 + (\Omega + 1)^3$$

$$= \left(\frac{1}{4}\right) \cdot (\Omega + 1)^2 \cdot (\Omega^2 + 4 \cdot (\Omega + 1))$$

$$= \frac{((\Omega + 1)^2 \cdot (\Omega + 2)^2)}{4}$$

$$= \left(\frac{(\Omega + 1) \cdot ((\Omega + 1) + 1)}{2}\right)^2$$

So the formula is also correct for $\Psi = \Omega + 1$. Proof is done.

Applications

I will solve two different real life problems using the Stirling Numbers of the Second Kind.

Problem 1: There are 7 different books that I want to wrap in gift wrap, and I want to divide them into 4 groups. In how many different ways can I do this?

Solution: This directly asks me to calculate S(7,4). Finding this is easy as I will use

S(ß, Φ) = S(ß -1, Φ -1) + Φ.S(ß -1, Φ)

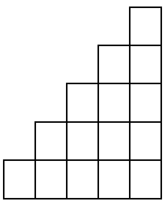
	ß	1	2	3	4	5	6	7
Φ		1	1	1	1	1	1	1
1			1	3	7	15	31	63
2				1	6	25	90	301
3					1			
4						1		
5							1	

	ß	…	$ß_0$-1	$ß_0$
Φ				
…		…	…	…
Φ_0-1		…	a	
Φ_0			b	b Φ_0 + a

We know S(3,1) = 1, S(3,2) = 3, S(3,3) = 1

Continuing the recurrence relation on the table above, we find that the answer is 301.

Problem 2:



This is a half chess board. On this board, in how mant way can I place three different rooks that do not threaten each other?

Solution: With K(Φ,r), I will show how I can do this on a half board with Φ x Φ dimensions.

Idea: On a half board with Φ x Φ dimensions, r non-threatening rooks can be placed S(Φ +1, Φ +1-k) ways. So S(Φ +1, Φ +1-r) = K(Φ,r).

Proof for the Solution: I will make an induction based on Φ.

If Φ =1, then K(1,1) = K(1,0) = 1, so S(1+1,1+1-1) = 1 = K(1,1); S(1+1,1+1-0) = 1 = K(1,0)

Assume that this is true for Φ =1,2,…, ß. I will prove it for Φ = ß+1.

K(ß+1,0) = K(ß+1, ß+1) = 1 for any 1≤ r ≤ ß.

So if there are no rooks on the bottom row of the half board, there are ßxß half squares left on the triangular board. On this, r rooks can be placed in K(ß,r) = S(ß+1, ß+1-r) different ways.

But if there are any rooks on the bottom row, then K(ß+1, r) = K(ß,r) + (ß+1) . K(ß,r-1)

K(ß,r) → If there are no rooks on the bottom row

(ß+1) . K(ß,r-1) → If there are any rooks on the bottom row (This rook has (ß+1) placement alternatives)

K(ß+1,r) = K(ß,r) + (ß+1) . K(ß,r-1) = S(ß,r) + (ß+1) . S(ß,r-1)

So there are (r-1) rooks on the (ßxß)-dimensioned half board above the bottom row. These can be placed there in K(ß,r-1) ways. After the (r-1) rooks are placed, for the rook on the bottom row, there are (ß+1)-(r-1) options left.

K(ß+1,r) = K(ß,r) + [(ß+1)-(r-1)] . K(ß,r-1) = S(ß+1, ß+1-r) + [(ß+1)-(r-1)] . S(ß+1, (ß+1)-(r-1)) is obtained by the Stirling Number recurrence relation.

This equals S(ß+2, (ß+1)-(r-1)), so K(ß+1,r) = S((ß+1)+1,(ß+1)-(r-1)) = S((ß+1)+1,(ß+1)+1-r), and this was what I intended to prove.

Putting in the numbers, K(5,3) = S(5+1,5+1-3) = S(6,3) = 90 is the number of ways I can place three different rooks that do not threaten each other on a 5x5 half chess board.

Conclusion and Evaluation

I reached my conclusions using two methods:

1- With Stirling Numbers (by deduction)
2- With induction

Induction was definitely short and easy. But it has a huge disadvantage: you have to know or at least guess the answer to do this. But using Stirling Numbers in deduction, this problem does not exist and you can find a formula without knowing the answer.

While doing research for this essay, I have become familiar with complex mathematical notations. I have learnt how to understand complex inductive and deductive proofs while trying to understand the source materials on the topic. I realized that breaking down complicated problems or equations into smaller and simpler parts and solving the easy, small components in a systematical way is a great approach to not only Mathematics, but every aspect of life in general, from workplace issues to school work. This topic has shown me that seemingly abstract concepts can be genuinely useful and horizon-expanding when applied to real life situations.

Stirling Numbers of the Second Kind are a mathematical concept related with permutations, combinations and the set theory. These numbers are most useful in mathematical proofs and solutions related to combinatorics, but it can also be applied to many other areas, such as the Bayesian View of the Poisson-Dirichtet Process in probability, the Approximation Theory and Finite Calculus Summations in calculus. It can even be used in computer programming for programs that contain highly sophisticated functions, such as Artificial

Intelligence Neural Network connection algorithms or Stock Market analysers that take combinations and statistics into consideration. It can also be used as a guide when solving simple daily problems involving partitioning, such as distributing cake to birthday visitors at a party or paying employees wages as an employer.

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